Zeno's Dichotomy: Undermining The Modern Response

Leo Groarke
University of Western Ontario

I

It was over 2,000 years ago that Zeno propounded his now famous arguments against motion. The four paradoxes he suggests allegedly lead to the conclusion that motion is impossible, and have sparked comments by philosophers as diverse as Aristotle, Hegel, and Russell. In the present paper, I will consider Zeno's first paradox--the Dichotomy--and the modern attempt to discount it. I will contend that the most philosophically interesting version of the Dichotomy undermines the widely held claim that the dilemma it presents can be resolved by an appeal to mathematical truths about the summation of an infinite series. If this is the case, then philosophers like Robinson,[1] Burnet,[2] Booth,[3] Russell,[4] Whitehead,[5] Carnap,[6] Quine,[7] Vlastos,[8] and others have failed to recognize the fundamental questions the Dichotomy raises. Nevertheless, their account of the paradox is the predominant one,[9] and has been expounded in books by Grünbaum[10] and Salmon.[11] Indeed, it has been claimed (by Maxwell and Feigl)[12] that the summation answer to Zeno's reasoning provides a paradigm solution to a philosophical problem, and (by Boyer)[13] that it demonstrates the power of the calculus. Within contemporary philosophy, it has also sparked the discussion of 'infinity machines' by Grünbaum, Vlastos, Black,[14] Thomson,[15] Benacceraf,[16] and others. The final outcome of this discussion has been the account of infinite sequences of acts adopted by philosophers like Grünbaum, Vlastos, Putnam,[17] and Boolos and Jeffrey.[18] If the position I argue for is correct however, then all of these philosophers--and philosophers (like C. D. Broad[19] and Philip Jones)[20] who apply similar considerations to Zeno's Achilles paradox--have failed to come to grips with the fundamental issues raised by Zeno's Dichotomy argument.

II

According to the Dichotomy argument, any motion is impossible because it requires the crossing of an infinite series of distances one by one. One half the total distance must be crossed, then one quarter, then one eighth, and so on ad infinitum. If d is the distance in question, then each distance in the series .5 d, .25 d, .125 d, . . . must be crossed in turn and this is alleged to be impossible. Zeno concludes that a motion which traverses d cannot be completed. We can summarize his argument as follows.

1) The completion of any motion requires the traversing of an infinite sequence of successive distances.
2) The traversing of an infinite sequence of successive distances cannot be completed.

Hence no motion can be completed.

In order to distinguish two interpretations of this argument, we need to consider two accounts of its second premise.

The usual interpretation of the Dichotomy claims that Zeno holds that the crossing of an infinite sequence of distances is impossible because it would require an infinite amount of time. When portrayed in this way, the paradox claims that the passage of an infinite number of time intervals (one for every distance in Zeno's sequence) must take an infinite amount of time, and that the motion across all the distances therefore requires an infinite amount of time. This version of the Dichotomy is easily unravelled, for--contrary to Zeno's supposed claim--an infinite series of time intervals may be completed in a finite time. To take an obvious example, the sequence .5 minute, .25 minute, .125 minute, . . . is infinite, but nevertheless elapses in one minute. It follows that this interpretation of the Dichotomy leads to the conclusion that its second premise is false, and to the conclusion that the argument it presents is not sound. As Quine writes:

> When we try to make this argument more explicit, the fallacy that emerges is the mistaken notion that any infinite succession of intervals of time has to add up to eternity. Actually when an infinite succession of intervals of time is so chosen that the succeeding intervals become shorter, the whole succession may take either a finite or an infinite amount of time. It is a question of a convergent series.[11]

Such a view (endorsed by Quine, Grünbaum, Salmon, Russell, Booth, etc.) has led to the claim that the Dichotomy is undermined by modern mathematics (i.e. because $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$). And whether or not this is the case, it certainly is true that, as Wesley Salmon claims:

> To whatever extent these paradoxes [the Dichotomy and the Achilles] raised problems about the intelligibility of adding up infinitely many positive terms, the nineteenth-century theory of convergent series resolve the problem.[22]

Yet it is a mistake to think that this is all there is to the Dichotomy paradox. Indeed, such a view ignores a philosophically more significant interpretation of the Dichotomy argument.

III

Because Zeno's own discussion of motion is not extant, it is difficult to settle exegetical questions about the various arguments he proposes. It should perhaps be pointed out that Aristotle's account of the Dichotomy does, for the most part, suggest the interpretation we have already noted. Nevertheless, Aristotle himself admits (at <u>Physics</u>, 236a) that this version fails to present all the issues the Dichotomy raises, and others have suggested that we can understand the Dichotomy in another way. For present purposes, it suffices to note that the alternative interpretation creates a more poignant philosophical dilemma (and one that undermines contemporary discussions of infinity). The account of the Dichotomy that Simplicius gives does back this second interpretation, but I will leave it as an open question which version of the argument was Zeno's. Suffice to say, it would be surprising if an argument which so naturally lends itself to the alternative interpretation was meant in the weaker way that Aristotle and most commentators suggest.

To reconstruct the Dichotomy argument, we need to reinterpret its second premise. According to the second version of the argument, the traversing of Zeno's infinite sequence of distances is not impossible because it requires an infinite amount of time. Rather, the crossing of the distances one by one is alleged to be impossible because the series contains no last element. In attempting to traverse the distances, one therefore crosses a particular distance and continually proceeds to another distance which must be traversed. One never reaches a last distance, its completion, and the consequent completion of the series.

It allegedly follows that one cannot complete the series.

A number of things must be said about this account of Zeno's reasoning. Those commentators who have confronted it have usually denied that it presents a genuine paradox. Philosophers like Grunbaum, Salmon, Vlastos,[23] and Thomson have maintained that it simply begs the question. They maintain that the completed crossing of Zeno's sequence of distances does not require the crossing of a last distance within the sequence. In his encyclopedia article on infinity, Thomson for example, writes that:

> If we turn to . . . the claim that an infinite sequence cannot be completed, we see that it begs the question. It is simply not true that however many of an infinite sequence we take, others remain, but only that whatever finite numbers be taken, others remain. What causes trouble is doubtless that (in the kind of sequence we are talking of) there is no last term, so that one does not see what finishing consists in in such a case. But, in the context of the Race Course [the Dichotomy] anyway, this trouble is illusory; finishing consists in occupying the limit point of the class . . .[24]

One must, of course, grant Thomson that "It simply is not true that however many of an infinite sequence we take others remain, but only that whatever finite numbers be taken, others remain". Yet this claim does not count against the Dichotomy argument, for the problem with our attempt to complete Zeno's sequence of distances is precisely that we cross the distances in the sequence a finite number at a time--i.e. <u>one by one</u>. The question at issue is how we can, proceeding in this way, ever come to a point where we have crossed an infinite number of distances, and the claim in question is the claim that we cannot do so because we never reach a distance which counts as the last element in an infinite sequence. The claim that we <u>can</u> cross an infinite sequence of distances despite its lack of a last element, and the bald assertion that we can thereby reach the limit point of Zeno's sequence, both beg the very question at issue. One needs an <u>argument</u> to establish that these claims are true, and Thomson does not provide one. Once one examines their consequences, it becomes clear that any such claims are wholly arbitrary.

One way to demonstrate the plausibility of the second version of the Dichotomy is to note that it derives its force from the following general principle.

> PSM (The Principle of Sequential Motion): The crossing of a sequence of successive spatial intervals does not take one to a point y unless y is reached by the crossing of one of the intervals in the sequence.

To gain an intuitive grasp of PSM, consider the motion of a pen that draws the line a b c d. Such a pen crosses the intervals ab, bc, cd, and the claim that in doing so it reaches some point outside the line ad would border on the preposterous. It is clear that the pen's crossing of the sequence of intervals ab, bc, cd does not, without a futher motion, take it to a point outside all these intervals. All this seems obvious and undeniable, and similar considerations show that PSM holds for finite sequence of spatial intervals generally.

The second version of the Dichotomy gains its force from the application of PSM to infinite sequences of spatial intervals.[25] For, given that the crossing of a sequence of intervals does not take one to a point outside the sequence, the crossing of the sequence Zeno proscribes can never take one to the endpoint of a motion (for this point lies outside all the intervals within the sequence). Any attempt to complete the motion by crossing the intervals one by one therefore seems bound to failure (though it is also necessary for the completion of the motion). It is as though one tried to move something to a point b by repeatedly moving it to points that all fall short of b.

When construed in this way, the Dichotomy argument is formidable. Its strength lies in the implausibility of the claim that PSM does not hold for all sequences of spatial intervals. This claim is particularly implausible given that PSM obviously holds for finite sequences of intervals. For what is the difference between finite and infinite sequences which would allow PSM to be transgressed in one case but not the other? As there appears to be no relevant differences, we must surely treat them similarly and apply PSM to infinite sequences of spatial intervals. Once we do so however, it immediately follows that one can never complete the one by one crossing of an infinite sequence of intervals. The crossing of the intervals in the sequence never brings one to a last interval, and never to a point where one has crossed all the intervals in the sequence. According to PSM, it follows that the crossing of the sequence cannot take one to such a point.

In their dealings with the Dichotomy argument, contemporary commentators have offhandedly rejected PSM by declaring that we can reach the endpoint of a motion by undertaking to cross Zeno's distances one by one. Yet the usual 'explanation'--that motion may be completed because it can be contained in a finite amount of time--does not address the problems that arise in regard to PSM. For even though the crossing of an infinite sequence of spatial intervals does not require an infinite amount of time, this does not, in any way, explain the violation of PSM that the completion of this crossing requires. In regard to finite sequences of motion, the crossing of a finite sequence of spatial intervals may (obviously) be contained in a finite amount of time, but this does not allow the violation of PSM. Hence there is no reason to suppose that it should allow the violation of the principle in regard to infinite sequences of spatial intervals. If commentators want to show that the completion of Zeno's sequence is made possible by its finite duration, then they must show that finite duration makes possible a transgression of PSM. This they have failed to do.

In view of such considerations, commentators like Grünbaum, Salmon, Thomson, et. al. simply beg the question when they claim that, despite its lack of a last interval, an infinite sequence of spatial intervals can be crossed one by one. And although this claim is, on the one hand, understandable (for motion is possible, and we can reach the endpoint of a motion), it fails completely as an answer to the philosophical problem Zeno's Dichotomy presents. PSM seems to be an obviously valid principle and one who rejects it needs to explain how it can be violated. Why is it that the crossing of a sequence of spatial intervals can, by itself, take one to a point outside all the intervals in the sequence? In completing the sequence one crosses the individual intervals and does nothing else besides. How then can one reach a point that one doesn't reach in crossing one of the intervals in question? And how is it that PSM holds in regard to a finite sequence of spatial intervals, but not in regard to all infinite sequences? One cannot answer such questions in a satisfactory way simply by ignoring them.

IV

In light of the problems the Dichotomy raises, some brief comments on contemporary discussions of infinite sequences of acts are in order. In connection with Zeno's paradoxes, questions about such sequences have centered around the possibility of 'infinity machines'--machines designed to perform an infinite sequence of acts. Machines designed to perform a variety of tasks (the counting of all the natural numbers, the execution of an infinite number of movements, etc.) have been discussed. Commentators like Grünbaum, Salmon, Putnam, and Boolos and Jeffrey have claimed that the possibility of such machines turns on the question of whether or not they can be designed to perform their sequences of acts within finite parameters (i.e. without employing an infinite amount of time, space, or energy). It should now be clear that there is more to the matter than this. For though the completion of an infinite sequence of acts would be impossible if it required an infinite amount of space, time, or energy, it does not follow that it is possible if this is not the case. In the latter circumstances it may be said to be impossible because such a sequence contains no last act. It seems to follow that an infinity machine would have to continually perform acts one after the other, without ever reaching a final act, its completion, and the consequent completion of the sequence in question.

In light of such considerations, it is a simple matter to base the conclusion that an infinite sequence of acts cannot be completed on reasoning analogous to that contained in the second version of the Dichotomy argument. Whereas the reasoning in the Dichotomy argument was built upon the principle PSM, this analogous reasoning can be grounded on the following principle:

> PSA (The Principle of Sequential Acts): The performance of a sequence of acts does not complete a particular task unless it is completed by the performance of one of the acts in the sequence.

Essentially, PSA stipulates that one cannot complete a particular task by performing (in sequence) acts which do not complete the task in question. Such a principle is plausible and once it is accepted it follows that an infinite sequence of acts cannot be completed, for no act within the sequence brings it to completion. It is as though one tried to accomplish a task by repeatedly performing actions which failed to complete the task in question.

In the present context, we need not consider PSA in detail. It is not difficult to make a case for it, and it straightforwardly follows that no infinite sequence of acts can be completed. In the present context, it suffices to note that the arguments that lead to this conclusion are analogous to the Dichotomy argument (once it has been interpreted in the outlined way). Until the problems the Dichotomy raises have been answered, there is little reason to expect the resolution of the more general problems that arise in regard to infinite sequences of acts. Until philosophers have a better answer to the Dichotomy paradox, it is premature to contemplate the completion of any such sequence.

NOTES

[1]John Mansley Robinson, An Introduction to Early Greek Philosophy (Boston: Houghton Mifflin, 1968). Robinson does not propose a solution to the Dichotomy, but his account of it demands the summation solution.

[2]John Burnet, Greek Philosophy: Thales to Plato (London: Macmillan, 1964).

[3]N. B. Booth, "Zeno Paradoxes," Journal of Hellinic Studies (1957).

[4]Bertrand Russell, Our Knowledge Of the External World (New York: W. W. Norton Inc., 1929).

[5]A. N. Whitehead, Process and Reality, (London: Macmillan, 1929).

[6]Rudolph Carnap, "Strawson On Linguistic Naturalism," in The Philosophy of Rudolph Carnap, edited by Schlipp (La Salle: Open Court, 1963).

[7]W. V. Quine, The Ways of Paradox (Cambridge: Harvard University Press, 1954).

[8]Gregory Vlastos, "Zeno of Elea," Encyclopedia of Philosophy (London: Macmillan, 1967).

[9]An alternative solution to the paradox is to be found in Aristotle's distinction between the actually and potentially infinite. A similar solution is proposed by Charles Chihara ("Completing An Infinite Process," Philosophical Review (1965)). I will leave a discussion of this attempt to refute the Dichotomy for elsewhere. I would argue that it too fails as an answer to Zeno's argument.

[10]Adolf Grünbaum, Modern Science and Zeno's Paradoxes (Middletown: Wesleyan University Press, 1967).

[11]Wesley Salmon, Space, Time, and Motion (Encino: Dickenson, 1975).

[12]Grover Maxwell and Herbert Feigl, "Why Ordinary Language Needs Reforming," in Richard Rorty (editor) (Chicago: University of Chicago Press, 1967).

[13]Boyer, The History of The Calculus (New York: Dover, 1949). Boyer writes that "The four paradoxes are, of course, easily answered in terms of the concepts of the differential calculus." (pp. 24-25).

[14]Max Black, Problems of Analysis (Ithaca: Cornell University Press, 1954).

[15]James Thomson, "Infinity in Mathematics and Logic," Encyclopedia of Philosophy; "Tasks and Super-Tasks," and "Comments on Professor Benacerraf's Paper," contained in Wesley Salmon (editor), Zeno's Paradoxes (Indianapolis: Bobbs-Merrill, 1970).

[16]Paul Benacerraf, "Tasks, Super-Tasks, and the Modern Eleatics," contained in Salmon (editor), Zeno's Paradoxes.

[17]See Hilary Putnam, "The Thesis That Mathematics Is Logic," p. 24, Mathematics, Matter, and Method (Cambridge: Cambridge University Press, 1975).

[18]See Boolos and Jeffrey, Comptability and Logic(Cambridge: Cambridge University Press, 1974).

[19]C. D. Broad, "Note On Achilles and the Tortise," Mind (1913).

[20]Philip Jones, "Achilles and the Tortise," Mind (1946).

[21]Quine, op. cit., p. 3

[22]Salmon, op. cit., p. 38.

[23]See Vlastos, op. cit., p. 372.

[24]Thomson, "Infinity In Mathematics And Logic," Encyclopedia of Philosophy, Vol. 3, pp. 187-188.

[25]For the sake of simplicity, we need only consider infinite sequences of ordinality in the present context.