



Variation Theory and Montessori Education

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Abstract: In this article we examine the relation between variation theory and Maria Montessori's didactic theory. Montessori believed that training and sharpening of the child's senses are crucial for their continued learning; she therefore developed specific sensorial materials to be used in Montessori preschools for such a purpose. As noted by interpreters of Montessori education, a key principle in this material, as well as in variation theory, is the use of variation and invariance. However, in this article, lessons in two different areas than the training of the senses are analysed from a variation-theoretical perspective on learning; these lessons originate from Montessori's own writings and from extracts from Montessori training courses. The result shows that a systematic use of variation and invariance can be seen as a more fundamental part of Montessori's didactic theory and is not only applied in the sensorial training. The article will offer theoretical concepts useful when explaining why lessons in various areas should be presented in the way they are described.

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Introduction

Montessori education is spread all over the world and the number of schools is constantly increasing (Ahlquist, Gustafsson & Gynther, 2011). Regardless of the part of the world or country the school is located in, visitors to Montessori schools will enter classrooms whose design is very similar. That is because the physical environment with its didactic¹ material is clearly described by Maria Montessori (e.g. Montessori, 1912; Montessori, 1914/65). Unlike the practical application, however, some interpreters of the pedagogy have noted that its theory is vaguely described in Montessori's own literature (e.g. Feez, 2007; Lillard, 2005; Montessori Jr., 1976/92). Feez (2007), for example, claims that Montessori only left behind what could be considered a practical application of the pedagogy rather than a theory.

However, a key principle in the application of the pedagogy, which has been noted by interpreters at a more theoretical level in recent years, is the use of variation and invariance (or contrast and sameness) within the training of the senses practised in Montessori preschools (e.g. Marton, 2006; Marton, 2015; Marton & Signert, 2008; Signert, 2012). Montessori believes that this training and sharpening of the child's senses is of great importance and even crucial for the child's continued learning since it will enlarge the field of perception and consequently offer a more solid foundation for intellectual growth (Montessori, 1948/93). This sensorial training, however, must be practised according to a certain principle in order to provide the right foundation for intellectual growth. Montessori describes it in the following words:

In the ordinary schools of today, teachers often give what are called 'object lessons' in which the child has to enumerate the various qualities of a given object: for example, its colour, form, texture, etc. But the number of different objects in the world is infinite, while the qualities they possess are limited. These qualities are therefore like the letters of the alphabet which can make up an indefinite numbers of words. If we present the children with objects exhibiting each of these qualities separately, this is like giving them an alphabet for their explorations, a key to the doors of knowledge. Anyone who has beheld not only the qualities of things classified in an orderly way, but also the gradations of each, is able to ready everything that their environment and the world of nature contains. (Montessori, 1949/82, p. 159)

Montessori's idea (1949/82) is to present didactic materials that demonstrate a distinct contrast between objects so that the differences between them are made obvious to the child. This will make the child curious and interested in exploring them. One of the didactic tasks of education is therefore to grade a series of objects which have to be identical with the exception of one single quality that has to vary. Consequently, the material is designed in order to help the child discriminate and classify among different sensorial aspects. One example of such material is the set of bells with which the child will distinguish and grade different tones. The bells are identical in appearance but differ with regularity in terms of tone. Consequently, what is common to all sensorial materials is that it is only the investigated quality that distinguishes two objects of a particular material from each other. In regard to other qualities, the materials are identical. Other sensorial materials designed in this way are, for example, the Brown prisms, all of which have the same length but differ only in the degree of thickness, whereby the child will learn to distinguish the thickest or the thinnest. The Red Rods, all have the same thickness but differ in length. The Cylinder blocks can be ordered by height by the child to distinguish tall from short, and with the Colour tablets the child will grade nuances of colour in order to distinguish between the darkest and the lightest. According to Montessori (1914/65), this contributes to the development of the child's language skills so that the child will be able to use their language in a more exact way. Children will be able to describe their experiences, for instance that a line is thin and not small (ibid.).

Montessori (1914/65) believes that her theory will have implications in the long term, as it develops the child's ability to recognize, observe, reason, judge and use the "power of discrimination". This is an important "psychic acquisition", which will retain their learning abilities. If the teacher prepares the objects of learning in an orderly way, the children's minds will enter "the Creation instead of the Chaos" (p. 130–131). Montessori (ibid.) gives a number of metaphorical illustrations to clarify what her didactic theory will accomplish, for example by exemplifying the difference between the scientist and a person without knowledge looking through the same microscope. The scientist will discover details which are impossible for an untrained person to see, which is also true of the astronomer who will see things clearly through a telescope compared with someone not familiar with that scientific field. Montessori

also compares a botanist and a visitor walking through a garden.

The same plants surround the botanist and the ordinary wayfarer, but the botanist sees in every plant those qualities which are classified in his mind and assigns to each plant its own place in the natural orders, giving it its exact name. It is this capacity for recognizing a plant in a complex order of classification which distinguishes the botanist from the ordinary gardener, and it is *exact* and scientific language which characterizes the trained observer. (Montessori, 1914/65, p. 130)

The aim of this article, however, is to explore, analyze and report on the validity of variation and invariance in other areas (and consequently other materials) than the training of the senses. The question we raise is whether the application of variation and invariance is valid in other areas as well and could therefore be seen as a fundamental idea in Montessori's view of learning that has not been noted so far. If so, a variation-theoretical perspective on learning could be seen as an important part of Montessori's didactic theory in general, thereby offering one answer to the question why lessons should be presented in the way described.

In the next section, we will initially describe some key concepts in variation theory. This section is followed by a description of the way in which teaching in Montessori education is implemented within two chosen areas at an elementary level. These descriptions are followed by analyses of the ways in which each description is related to a variation-theoretical perspective on learning. The article ends with a discussion of the results and their practical implications.

Learning to see in order to learn to do - a variation-theoretical perspective

According to Marton (2015), a distinction can be made between two ways of learning in school.

One way is to make the object of learning (that which is to be learned) your own, to discern the important aspects of the content of learning and the relations between them. The other way is to learn what to do and say in order to meet the demands imposed upon the learner by the teacher or the test. (p. 14)

If the latter kind of learning is stressed, less of the first kind might happen. Hence the teacher should above all

create conditions which will allow the students to acquire the necessary aspects of the object of learning and the relationships between them. In that case, students will learn how to do things by seeing how things are related to each other, rather than just learn a certain order as told by the teacher. This is significant for a variation-theoretical perspective on learning which indicates that "mastering an educational objective amounts to discerning and taking into consideration its necessary aspects" (ibid., p. 23). Thus in a variation-theoretical perspective learning is seen as "learning to see" (ibid., p. 36). According to Montessori (1914/65), this is precisely what working with the sensorial materials seeks to establish within the child. As Montessori points out:

The child then has not only developed in himself special qualities of observation and of judgement, but the objects which he observes may be said to go into their place, according to the order established in his mind, and they are placed under the appropriate name in an exact classification. (Montessori, 1914/65, p. 129)

When learning is seen as "learning to see", it follows that someone has learnt something when he/she is aware of other or more aspects of a phenomenon than before (Marton & Booth, 1997). Learning is therefore "mostly a matter of reconstituting the already constituted world" (ibid., p. 139). However, when we experience a phenomenon, we often find it unclear, so "the whole needs to be made more distinct, and the parts need to be found and then fitted into place, like a jigsaw puzzle that sits on the table half-finished, inviting the passerby to discover more of the picture" (ibid., p. 180). What Marton and Booth say here is that the relationship between what can be seen as the whole and its parts must be visualized if learning is to be made possible. As Lo (2012) argues: "There must be a whole to which the parts belong before the parts can make sense to us. We cannot learn mere details without knowing what they are details of. When the whole does not exist, learning will not be successful" (p. 26). This is also pointed out by Montessori, who formulates it as follows: "to teach details is to bring confusion; to establish the relationship between things is to bring knowledge" (Montessori, 1948/96, p. 58). Montessori also points out the importance of classification, for example when a child is about to study living beings. Classification of animals then gives the child a picture of the great number of

animals as well as their diversity. This will help the child to distinguish between the different groups of animals and from there go into details (ibid.). If the learner has not seen a specific necessary aspect or part in relation to what can be seen as the whole, and therefore not made the object of learning her own, it is, according to variation theory, seen as “critical” (Lo, 2012; Marton, 2015). This means that it has to be discerned by the learner in order to meet the educational objective. The teacher’s ability to help the learner to do this will, of course, be facilitated if the teacher is aware of the critical aspects of a certain learning object and is thereby able to direct the learner’s view towards such aspects. Montessori is critical of traditional² education where the teacher talks and the child remains passive. According to Montessori, the child does not learn by just listening to words; the child has to make discoveries. To consider mind and movement as separate from higher functions is one of our times “greatest mistakes”, states Montessori (1949/73, p. 140). Instead, Montessori regards mind and body as one entity. This kind of standpoint implies a different school environment, organized with materials that allow children to make their own discoveries (Ahlquist, 2012). In Montessori education, the Montessori materials serve such a purpose. This does not mean leaving children alone while working with the material. The teacher’s responsibility is to observe the children’s work without interfering, letting them instead work at their own pace, supporting them when needed and challenging them by discussing and examining their discoveries and letting them express their understanding.

Variation and invariance

According to variation theory, the learner has to be aware of the difference between at least two features in order to discern them. Marton (2015) gives as an example that a prerequisite for the inhabitants of an entirely green world to be aware of its “greenness” is that they are exposed to a different colour as well. As Marton argues “You cannot start with two green things and thus become aware of the colour green. Nor can you understand what truth is by inspecting true statements only” (ibid., p. 48). Rather, learners have to discern what green is at the same time as they discern what is not green. This is, of course, possible only if green is exposed in contrast to a different colour. However, this is not enough. In addition to being exposed to varied colours so that they will be contrasted with each other, other aspects like shape and size have to be kept invariant in order to

make it likely that the aspect in focus (colour) will be discerned.

Generalization and fusion

Once the learner has found the meaning by contrast, he/she has to generalize the aspect which has previously been separated. If the aspect, for instance, is colour, generalization is achieved by keeping the colour invariant but varying other aspects such as form and size. The aim of generalization is not to find out what different aspects have in common; rather, it is to find out how different aspects vary. If the aspect is colour, the conclusion we will draw through generalization will therefore be something like: “so this can be red, and this and this”, rather than “they are all red”. As Marton (2015) points out: “Through contrast, we are trying to find necessary aspects of the object of learning, those that define it. Through generalization, we want to separate the optional aspects from the necessary aspects” (p. 51). However, from a variation-theoretical perspective, it is important here to emphasize that such generalization should always be preceded by contrast (ibid.).

The final step is to let the learner experience simultaneous variation in all relevant aspects. In variation theory, this pattern of variation is called fusion: “it defines the relation between two (or more) aspects by means of their simultaneous variation” (Marton, 2015, p. 51). The learner will then experience simultaneous variation in all relevant aspects. In the case of colour, the learner will, for instance, experience that any colour might appear with any form.

Variation theory in other areas than sensorial training

Initially, we stated that Montessori, as in variation theory, emphasized that the child will develop their ability to “see” in the work with the sensorial materials in preschools by using patterns of variation and invariance. We will now look into the ways in which certain other areas are dealt with according to Montessori at an elementary level and whether it can be assumed that Montessori designed the materials and the teaching with such a purpose in other areas as well. We have chosen to look into one specific area in teaching arithmetic and one in teaching geometry. We decided to choose these areas as they are either described in detail in Montessori’s literature or in oral presentations within Montessori training.

Introducing numbers

When the teaching of numbers is introduced in Montessori education, teachers use a material called Number Rods, shown in Figure 1, which consists of ten rods of different lengths. The shortest is one decimetre long, the longest one metre, while the intervening rods are divided into sections one decimetre in length. These sections are coloured alternately red and blue.

Figure 1

The Number Rods. Photo by Eva-Maria T. Ahlquist.



In Montessori's description (1914/65) of how the material is supposed to be used by the teacher she writes:

When the rods have been placed in order of gradation, we teach the child the numbers: one, two, three, etc., by touching the rods in succession from the first up to ten. Then, to help him gain a clear idea of number, we proceed to the recognition of separate rods by means of the customary lesson in three periods. We lay the three first rods in front of the child, and pointing to them or taking them in the hand in turn, in order to show them to him we say: "This is one." "This is two". "This is three". We point out with the finger the divisions in each rod, counting them so as to make sure, "One, two: This is two." "One, two, three: This is three." Then we say to the child: "Give me two." "Give me one." Give me three." Finally, pointing to a rod, we say, "What is this?" The child answers, "Three," and we count together: "One, two, three." (1914/65, p. 170)

When the children have worked with the rods for some time, the teacher will introduce the Sandpaper Numbers, which consists of a box with cards on which the numbers from one to nine are cut out in sandpaper. Montessori (1912/64) now states that the child is supposed to touch the numbers in the direction in which they are written and to name them at the same time.

He/she is also shown how to place each figure upon the corresponding rod. After working with the rods and numbers, the teacher will introduce the Counting Boxes shown in Figure 2. This material consists of a box divided into ten compartments (0–9), on each of which the corresponding number is printed, and the child places the correct number of pegs in the compartments (Montessori, 1934).

Montessori also writes that another exercise associated with the child's work with the boxes is to put all the Sandpaper Numbers on the table and place the corresponding numbers of cubes, counters and the like below (ibid.).

Figure 2

The Counting Boxes. Photo by Eva-Maria T. Ahlquist.



The didactic material used for teaching the first arithmetical operations is the same one as used for numeration, the Number Rods. Montessori (1912/64) writes:

The first exercise consists in trying to put the shorter pieces together in such a way as to form tens /.../ In this way we make four rods equal to ten. There remains the five, but turning this upon its head (in the long sense), it passes from one end of the ten to the other, and thus makes clear the fact that two times five makes ten.

These exercises are repeated and little by little the child is taught the more technical language; nine plus one equals ten, eight plus two equals ten, seven plus three equals ten, six plus four equals ten, and for the five, which remains, two times five equals ten. At last, if he can write, we teach the signs plus and equals and times /.../ When all this is well learned

and has been put upon the paper with great pleasure by the children, we call their attention to the work which is done when the pieces grouped together to form tens are taken apart, and put back in their original positions. From the ten last formed we take away four and six remains; from the next, we take away three and seven remains; from the next, two and eight remains; from the last, we take away one and nine remains. Speaking of this properly we say, ten less four equals six; ten less three equals seven; ten less two equals eight; ten less one equals nine. In regard to the remaining five, it is the half of ten, and by cutting the long rod in two, that is dividing ten by two, we would have five; ten divided by two equals five. (p. 333–334)

Analysis of how numbers are introduced

Initially, we can note that the material presented above, in itself, isolates the quality “number” by its design. When the numbers 1, 2, 3... are introduced, it is only the numbers that vary. Other qualities in the material are identical. Furthermore, “one” is introduced in contrast to “two” and “three” and so on.

Another important aspect when it comes to the design of the lessons is the order in which these lessons are given. Looking at the sequences of the lessons, it seems clear that the purpose of such sequences is to make it possible for the child to initially find out the meaning of numbers by contrast and then, later, generalize the aspect which has previously been separated. This, for example, is done by working with different objects such as counters, cubes and the like. which the child matches with the Sandpaper numbers or the right compartment in the Counting boxes.

The importance of contrast is also evident when arithmetical operations are introduced with the Number Rods. In Montessori’s description of how this should be done, it is noticeable that addition is introduced in contrast to subtraction and that multiplication is introduced in contrast to division. The contrast between addition and subtraction, for example, is made by first putting rods together and then, later on, taking them apart. In this way it is possible for the child to “see” the relationship between, for example, $3 + 2 = 5$ and $5 - 2 = 3$. When Montessori links addition and subtraction together in this way, the relationship when it comes to what can be seen as parts and wholes is stressed, which may make addition easier to grasp since it is introduced in contrast to subtraction.

When comparing the work with the Number Rods and the Counting boxes, it might seem at first sight as if the children in their work with Counting boxes repeat the same exercise as with the rods. However, we have to look at the way the Number Rods and the Counting boxes are designed. If we say that the number that each rod corresponds to can be seen as “solid”, we then have to say that the pegs in the Counting boxes can be described as “loose”. This corresponds to two critical aspects, the ordinal and cardinal property of numbers, which the child has to “see” in order to grasp the rules of arithmetic. Ordinal property means that each number refers to a place in an order (1st, 2nd, 3rd...). Cardinal property refers instead to the “manyness” of things (one book, two books...). Both aspects can be noticed in the way the work with the Number Rods and Counting boxes is designed, but each material stresses different aspects. When the children are working with the rods, they grab “the manyness”, or as Montessori (1934) describes it, “one united whole”, that the rod in itself represents in their hands, even if they will also be able to identify the ordinal property when, for example, counting each section of the rod. The same can be said about the work with the Counting boxes, but in this case the ordinal property is more prominent when counting each peg than in the work with the Number Rods, even if the main aim of the work is to match each compartment with the right number of objects.

What can be seen as an additional critical aspect when handling the Number Rods as described above is that numbers are wholes that can be divided into parts. This may be noticed by the child in the work with arithmetical operations. When a child, for example, tries to put rods together in such a way that they form tens, this will illustrate that wholes can be divided into parts. In this example, the work done by the child illustrates that ten can be split into nine and one and that they are parts of the whole ten and so forth.

Introducing triangles

Geometry is presented in preschool by providing children with sensorial experiences and presenting the names of the different geometrical objects. Montessori argues:

Observation of form cannot be unsuitable at this age; the plane of the table at which the child sits to eat his soup is probably a rectangle; the plate which contains the meat he likes is a circle; and we certainly do not consider that the child is too immature to look at the table and the plate. (1948/83, p. 165)

The geometry material in preschools consists of blue Geometric solids containing objects of ten different shapes, a Geometry Cabinet with thirty-six plane figures and Triangle boxes used to construct polygons. These materials are also utilized in elementary education. This is, in fact, something that is fundamental in the Montessori curriculum: materials from preschool build the basis for further studies at higher levels. “They [the materials] form a long sequential chain of learning: each material can be placed within a hierarchy in which the simplest one forms the basis for the next. Nothing is left to chance in this sequence, everything is provided...” (Tornar, 2007, p. 120).

At an elementary level, there are more materials than mentioned above. Here, though, we will focus on the work with the Geometry Cabinet and how it is used to make it possible for the children to deepen their knowledge of triangles. The study of geometry in elementary classes is a work of experimentation and discoveries. Here we present extracts from the introductory notes to geometry from the AMI course in Bergamo:³

Montessori’s psycho-geometry reveals the essential place that geometry holds in human development, both historically and now, in the educational system. Psycho-geometry seeks to show the geometry inherent in life: organic and inorganic nature.

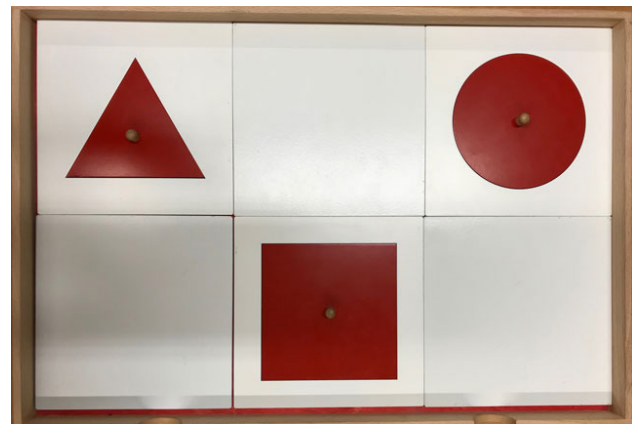
For example, inorganically: crystals, snow-flakes and organically: formation of flowers, molecules etc. Further, we look at the supra-nature, the work of humans in constructive architecturally and in other designs. Similarly, it can be seen that geometry is based upon the observable order of our world. Geometry, therefore, cannot be seen only in the abstract. One can study geometry by studying the historical evolution of humans and also by observing carefully the world in which we pass our daily lives./.../ Geometry, γεω; geo- “earth”, μετρία -metron “measurement”, the measurement of the Earth on which we live. This implies the relationship between humanity and the objects of our Earth, as well as knowledge of the relationship between these objects themselves. We study fundamental elementary Euclidian geometry./.../Our [the Montessori] geometry is made up of a) plane geometry, the study of the properties and relations of plane figures, and b) solid geometry, the study of figures in space, figures whose plane sections are the figures we have already studied in plane geometry.

In this article we will focus on the work with the Geometry Cabinet and how it is presented so as to expand the children’s knowledge of the different shapes. Here, we will concentrate on different types of triangles. At the elementary level, the geometry lessons, when adequate, will relate to the history of the subject area, and the etymology of words will be identified for each new concept the children meet. The study of triangles shown below will focus on the triangle examined by its side and by its angles and the work on uniting the sides and the angles.⁴ The study of other plane figures is largely similar to the work with triangles.

The Geometry Cabinet consists of six drawers, each containing six wooden squares with geometric plane figures in the same colour⁵ inserted in each square. On top of the cabinet there is a presentation tray shown in Figure 3, representing three of the geometric figures that will be found in the cabinet. The tray has six wooden squares, three of which contain an equilateral triangle, a square and a circle.

Figure 3

The presentation tray. Photo by Eva-Maria T. Ahlquist.



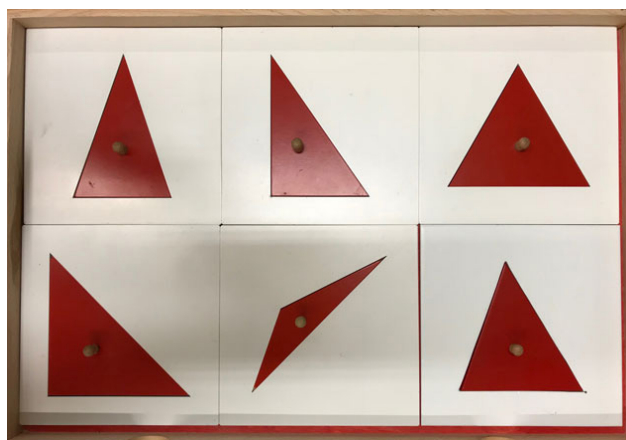
Each figure in the cabinet has a small handle in the centre, making it possible to lift up the figure when taking it out of the frame. The first drawer, shown in Figure 4, contains six triangles with the following shapes; an equilateral triangle, an acute-angled scalene triangle, an acute-angled isosceles triangle, an obtuse-angled isosceles triangle, a right-angled isosceles triangle and a right-angled scalene triangle.⁶

The second drawer has six rectangles, all with the same height, ten centimetres, and increasing from five centimetres in length to ten centimetres (the last one representing a square). The third drawer has six circles where the diameter increases from five to ten centimetres.

The fourth drawer has regular polygons from a pentagon to a decagon and the fifth drawer has other quadrilaterals, such as an irregular quadrilateral,⁷ trapezium,⁸ an isosceles trapezium, a kite,⁹ a parallelogram and a rhombus. The last drawer has five¹⁰ curvilinear figures, two kinds of quatrefoils, a curvilinear triangle,¹¹ an oval and an ellipse and also an extra triangle (an obtuse-angled scalene triangle).

Figure 4

The first drawer. Photo by Eva-Maria T. Ahlqvist.



The children should be familiar with the name triangle and the etymological origin and be asked to pick out the triangle among other polygons from the cabinet and identify triangles by going out in nature or visiting the city.¹² Subsequently, the teacher introduces different types of triangles in the first drawer of the cabinet. First, the three triangles on the upper row are examined by its sides. The teacher presents the scalene and the isosceles triangle by having the two triangles stand in an upright position on the base, the scalene with “limping” legs and the isosceles with a pair of legs of equal length. Then these two triangles are compared with the equilateral triangle, whose sides are of equal length. The children can observe this by rotating the triangle in its frame. Then there will be a repetition of the names used, performed as what Montessori (1912/64, 1914/65) calls a three-period lesson. This means that after the teacher has given the presentation above, he/she checks if the children are able to recognize the different types of sides; and finally, the children confirm their understanding by naming and describing each triangle.

The next step is to examine the angles of the triangles placed on the bottom of the drawer, starting with the right-angled triangle, with the right angle as a base letting

one of the legs follow the base and the other pointing upwards. The children compare this right angle with the angle between the floor and the wall in the classroom. The teacher tells the children the name of the angle. The next triangle explored is the scalene. The teacher asks the child to compare the scalene angle with the right angle in order to discover the difference. The children will then be asked to compare the obtuse angle with the acute angle by letting the child touch both of them. Then the teacher asks the children to examine all three angles of the acute-angled triangle, discovering that all angles are acute. The same procedure is done with the right-angled and the obtuse-angled triangle.

The third step is to unite the sides and angles. The teacher asks the child to write labels with the names of the sides and labels with the names of the angles of all six triangles. Each triangle will have two labels. Then the children are asked to tear off the word triangle from the labels and then unite the words of the angles (for instance, acute-angled) and the words of the sides (for instance, scalene). Finally adding the labels on which the word triangle is written (here exemplified by the acute-angled scalene triangle). There is then a discussion about the equilateral triangle: Should the triangle be called equilateral triangle or “equiangular” triangle? The children are asked to look for the name commonly used and will choose the name equilateral. The labels are rewritten on an undivided label for each triangle.

The children now order the triangles by constructing a coordinate system with two axes. On one of the axes the children put the word Angles written on a label, and below three labels with the names of the angles. On the other axes, the children put the word Sides, and below the names relating to the sides. The coordinate system will in this way have nine spaces, and the child is asked to put the triangles in their right positions. When this is done, there will be three empty spaces. The children now have to explore if there are triangles missing which could be placed in the coordinate system. By constructing triangles with help from The Box of Sticks¹³ shown in Figure 5, they will discover that there should be an obtuse-angled scalene triangle (which can be found in the last drawer of the cabinet), but it is not possible to construct a right-angled equilateral triangle or an obtuse-angled equilateral triangle.

Analysis of how triangles are introduced

Montessori argues that the child has to have embodied experiences in order to distinguish different

Figure 5

The Box of Sticks. Photo by Eva-Maria T. Ahlqvist.



shapes and she criticizes the traditional way of teaching as it does not help the child to recognize and remember the shapes.

The teacher draws a triangle on the blackboard and then erases it; it was a momentary vision represented as an abstraction; those children have never held a concrete triangle in their hands; they have to remember, by an effort, a contour around which abstract geometrical calculations will presently gather thickly; such figure will never achieve anything within them; it will not be felt, combined with others, it will never be an inspiration. (Montessori, 1917, p. 270)

Montessori education combines movement and language. This is an essential feature of Montessori's didactic concept since manipulating an object facilitates the possibility to isolate the quality of the object in question. When starting by examining the different triangles, the fundamental condition is that the child already knows what characterizes a triangle. This was

done with the presentation tray, where the triangle was initially contrasted with the square and the circle. What varies is the shape since the colour is invariant. In accordance with variation theory, the foundation of meaning here is the difference in shape. If instead the teacher had picked out three triangles of different colours, one blue, one red and one green, and told the child that all of them are triangles, the child would have had difficulty in grasping what a triangle is because there were no alternatives to a triangle. And even if there had been different geometrical shapes, but all of different sizes and in different bright colours, it would, according to variation theory, have been problematic for the child to focus on the essential aspect. As Feez (2008) states, the Montessori materials might seem to be old-fashioned in comparison with materials designed today, which often (p. 209) "interweave elements of educational knowledge with fantasy, popular culture and child-oriented imagery". But in accordance with Montessori education, the materials are learning-oriented as there are no distractions from what is to be focused on. When the child can distinguish the triangle among the other shapes in the presentation tray, a generalization is made by identifying a variety of triangles as triangles, regardless of their size, colour, rotational orientation or type. In the latter case, the child will not only discern the three corners of the triangle in order to recognize it as a triangle but now he/she also has to learn to discern the characteristics which are not necessary aspects of the learning object (such as size, orientation and so on). This order of sequence, in accordance with what is emphasized in variation theory, means that generalization is preceded by contrast. The next step is to examine critical aspects of the triangles: the sides and the angles. Examining the sides is made by contrasting the scalene triangle with the isosceles and so on. The child does this by holding the triangles in his/her hand, which allows twisting and turning the different figures. This allows the child to internalize the shape, even when it is put in different positions. The same procedure is done by contrasting the angles. The child can insert the right-angled triangle in a corner and contrast it with an acute angle or an obtuse angle. By contrast, the child will be able to discern the necessary aspects of the object of learning. This again is followed by generalization, where the child has to identify either the different sides and in another exercise, identify angles among triangles that differ in many qualities. By this generalization the child is able to separate different aspects from the necessary aspects.

When the child is able to identify the sides of triangles and knows what characterizes their angles, the two qualities are united in one and the same triangle. This act can be seen as what Marton (2015) refers to as fusion. This exercise is done by organizing the different types of triangles as a pattern in a coordinate system. During this exercise, the child can use the Box of Sticks as an aid to construct the different triangles. As the lengths of the sticks differ in colour, the child will easily pick the correct length of the side of the triangle and by using a “guide angle” (a right angle) they will experience that every angle smaller than a right angle is acute, as well as that every angle larger than a right angle is obtuse. This work will help him or her to make certain observations, for example that all triangles have acute angles, but in order to be called an acute-angled triangle, all three angles have to be acute. They will also be able to realize that two types of angles, right and obtuse, can be combined by two types of sides, but the acute angle can be combined with all three types of sides. This exercise, which has been completed by fusion, where the child has combined and defined two critical aspects by a process of their simultaneous variation, makes it possible for the child to experience that there are only seven types of triangles.

Discussion

The activities within the areas described above are the result of Montessori’s empirical research on how children learn. As shown in the analyses, the use of variation and invariance is to the fore in those activities. However, the latter is not made explicit by Montessori in her literature except for the sensorial training described in our introduction. In Montessori’s (e.g. 1912/64, 1914/65, 1948/83) descriptions of the materials and their use, she mainly deals with the didactic questions ‘what’ and ‘how’ rather than explicitly expressing why the content should be treated in the way it is described. Cossentino (2009), who has examined Montessori teacher training courses, points out that this is also significant for the way the training is conducted by tradition and therefore sees it as a “transmission” of technique, rather than a development of an understanding of why the material should be handled in a certain way. When there is a lack of such competence, it is more likely that the presentation with the Montessori materials will be performed in an instrumental way. It is also reasonable to assume that the participants are poorly equipped for teaching in areas which have not been dealt with in their training. In a study conducted by Gynther (2016), one of the Montessori teachers did not know how

to introduce the concept of proportionality as it had not been covered during her Montessori training. If she had understood Montessori’s systematic use of variation and invariance as part of the didactic theory, it is reasonable to assume that she would have been able to clarify what is proportional, as well as what is not proportional when the concept was introduced to the children. The point we are making here is that Montessori training must not only make participants very familiar with the Montessori materials; it must also develop their awareness of the underlying theory in order to discern the why in the practical application and hence be prepared for the work to come. The theoretical concepts presented here will also function as a platform for teachers and others when reviewing the ways in which different topics are treated within various Montessori environments.

Our analyses show that the theory behind Montessori’s didactic material, due to the design of the material and how the lessons should be given, is supported by variation theory, and we reveal that Montessori has clearly searched for and identified what in variation theory is referred to as critical aspects. Montessori’s (1948/83) own observed lessons in which such identification is not done by the teacher further reinforces this result. Montessori describes, for example, a teacher who was asked to show how to present two plane figures, a square and a triangle, by teaching the child the shape of the figures. The teacher handed out the square and made the child touch the outlines while she said “This is one line, another, another, another; there are four, just count with your fingers how many there are. And the corners, count the corners, feel them with your finger, press on them, there are four of them too. Look at it carefully; it is the square” (ibid., p. 109–110). According to Montessori, the teacher was not presenting the shape of the square; she gave the children the idea of other concepts, sides, angles and numbers. Montessori states that this was an abstract lesson as a side or an angle cannot exist without relating to the whole figure, in this case the square, and in addition the child had to be able to count to four. Without knowing how to separate the shape of a square from another shape, and instead make a mathematic analysis, the lesson will only confuse the child. Montessori asserts that it is necessary that the teacher knows how to give a lesson. What she wants to make clear is that children can distinguish the shape of the figures by simply contrasting them.

What Montessori implies by replicating lessons like the one described here is that the critical aspects must be identified by the teacher if the necessary conditions

for learning are to be created. This is in accordance with Marton (2015), who declares that the design of the lesson must reflect “the ways of seeing something we are trying to help the students to develop” (p. 256), that is, what it is intended that the student should learn. Furthermore, such identification seems crucial if Montessori teachers are to be able to succeed in their efforts to observe and follow each child as the pedagogy advocates. Of course, this is because, if teachers are aware of the critical aspects of a learning object, it will be much easier for them to identify by observation if the intended object of learning has been reached, alternatively what aspect the learner is not yet able to discern. That being said, we want to make clear that although Montessori specifies aspects that must be considered, the teacher has to identify what is to be regarded as critical in *each* child’s meeting with the learning object. What is regarded as critical could thus differ between children, which is why Montessori emphasizes the need for teachers to observe (Montessori, 1912/64, 1948/83, 1949/82) in order to be able to respond to children’s expressed understanding as well as to challenge children’s knowledge development.

As the use of variation and invariation is not always clearly expressed in Montessori’s literature, even if the material and the sequences of lessons are described in detail, we believe that this article will have an impact on Montessori education. We also believe that it can contribute to variation theory with the idea that not merely seeing helps children to make knowledge their own. The fact that children are given the possibility to discover critical aspects by grasping them must be regarded as crucial. As Montessori (1934/2011) says, activities “involve the hand that moves, the eye that recognizes and the mind that judges” (p. 11). Viewing the body and the mind as interwoven (Ahlquist, 2012) in the explorative work accomplished by the children, as shown in the analyses above, supports the use of variation and invariation. With such an approach, the teacher should reasonably be able to assume that the object of learning has given the children an understanding and that the knowledge has become their own.

Notes

With regard to didactics we refer to the basic questions: *What* is it that should be taught and *how* should it be made available for the learner? These questions also include a “why”: *Why* should something be taught and *why* should it be taught in a certain way? The *how* includes the learning resources, in this case the

Montessori material, guided by the question: *Through what* do we learn? For further reading, see Liberg (2012) and Jank & Meyer (2018).

1. Traditional here refers to a way of teaching in which the children have few opportunities to make experiences of their own. Rather, what is to be taught is mainly “transmitted” to the child by the teacher (Granberg 2014).
2. Extracts from personal notes by Ahlquist from the AMI, Associazione International Montessori course, 1981–1982.
3. Here we concentrate on just one section of the study of triangles. The Montessori material in geometry consists of other materials, such as the Constructive Triangle Boxes, the Box of Stars, the Metal Insets and the Yellow Area Material.
4. Montessori suggested that the geometric figures should all be blue and the bottom of each tray should be yellow. Some manufacturers of the material made the geometric figures red and the bottom of the tray white.
5. The last of the seven types of triangles, the obtuse-angled scalene triangle, is found in the sixth drawer.
6. In American English, it represents a trapezium.
7. In American English, it represents a trapezoid.
8. This is a special kind of trapezium as there are two pairs of sides of equal length or all four sides of equal length but none of the sides are parallel. The drawer could also contain a boomerang, depending on the manufacturer.
9. Some manufacturers include a third quatrefoil (an epicycloid). In those cases, the drawer contains six curvilinear figures.
10. Also known as the Reuleaux triangle.
11. Examples of such work are given in Ahlquist, Gustafsson & Gynther (2018).
12. The Box of Sticks contains sticks from one unit to ten units, each unit in a different colour. Every stick has a hole in each end in order to be able to unify them with each other when constructing geometrical shapes. There are also neutral sticks with units from one to ten but of different lengths compared with the coloured sticks, as they represent irrational numbers. The material also consists of a set square, which is used to identify the angle as a right angle.

Competing Interests

This manuscript has been peer-reviewed externally and the process was anonymous. The final decision was made by the Associate Editor Christina Gustafsson.

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