A Computer Method for Four-Variable Trend Analysis Illustrated by a Study of Oil-Gravity Variations in Southeastern Kansas

By John W. Harbaugh

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ABSTRACT

A method for fitting four-variable trend hypersurfaces by least squares has been programmed for the IBM 7090 computer. The program fits first-, second-, and abbreviated third-degree hypersurfaces to irregularly spaced data. The program automatically contours the intersection of each hypersurface with a block whose top, bottom, and four sides represent planes located in three-dimensional space. This permits the four-variable or four-dimensional hypersurfaces to be visualized. The program also automatically plots original data and residual values in a series of horizontal slice maps. The theory and operation of the program are discussed and illustrated in detail.

The program has been used to interpret variations in crude oil gravity from place to place and in different Paleozoic stratigraphic horizons in southeastern Kansas. Hypersurfaces were fitted to API oil gravity as a function of geographic location and depth below the surface. The four variables involved are (1) API gravity, (2) well depth, (3) north-south geographic coordinates, and (4) east-west geographic coordinates.

The trend hypersurfaces, distribution of residual values, and other considerations suggest that oil-gravity variations in southeastern Kansas have been affected by both well depth and environment of deposition. The tendency for API gravity to increase with depth is complicated by regional effects that may reflect differences in environment of deposition. The result is an overall increase in API gravities in a west-northwest direction. Of interest is a tendency for residual API gravity "highs" and "lows" to be clustered in certain geographic areas even though oils from different stratigraphic zones are involved. This, in turn, suggests that the depositional environment may have affected oil gravities in a given locality much the same way from one geologic period to the next.

The computer program described in this report may have a number of geological applications, and can be used readily by anyone having access to an IBM 7090 or 7094 computer.

INTRODUCTION

This report deals with a method for using an IBM 7090 or 7094 computer for fitting four-variable trend surfaces to geologic data. One of the purposes of this report is to emphasize the potential usefulness of this method in interpreting certain types of geological information. Krumbein (1956, 1959) has outlined the principles of three-variable trend surface maps and Peikert (1962, 1963) has illustrated the techniques of four-variable trend surfaces in interpreting specific gravity variations in intrusive igneous rocks. A second purpose is to demonstrate the use of the method with an example based upon variations of API gravity of crude oil in southeastern Kansas. A third purpose is to present the details of the theory and operation of the computer program. It is suggested that the program might profitably be used in oil exploration and in other geological problems. The program is a modification of a program developed previously by the author (Harbaugh, 1963).

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Geologists have long been concerned with trends. Some geological trends are readily shown on maps by contour lines. For example, a structure contour map portrays a three-dimensional surface in which two of the dimensions are "geographic" and are represented by the length and breadth of the map. The third dimension is the elevation of the surface represented by the contours. Thus, the surface may be said to be embedded in three-dimensional space.

It should be pointed out that, from a mathematical viewpoint, the terms "variable" and "dimension" may be used somewhat interchangeably. A surface that occupies three-dimensional space may be considered to represent a mathematical function involving a total of three variables. We can readily graph mathematical functions of two or three variables, using two or three dimensions. On the other hand, we can also deal mathematically with functions of four or more variables, but we have difficulty in graphically representing spatial relationships in four or more dimensions.

FOUR-DIMENSIONAL SURFACES

One of the objectives of this report is to emphasize that geologists commonly deal with relationships which may be thought of in a four-variable or four-dimensional sense. Consider the problem of the distribution of pores in a rectangular block of rock. All rocks are porous, and, therefore, at every point within this block, some particular value of porosity exists. Because porosity is a variable, and because we may regard a variable as a dimension, in a sense we are dealing with four dimensions if we consider the spatial distribution of pores in the rock.

Visualizing the fourth dimension poses a problem. We can, however, represent a fourth variable in three-dimensional space by simply plotting the particular values of the variable at the points where they occur in a three-dimensional coordinate system. In Figure 1, the three axes of a coordinate system are represented by the variables *w*, *x*, and *y*. The fourth variable, *z*, cannot be graphically represented by an axis, but can be represented by values at different points, the two points, z_1 and z_2, being shown for illustration's sake.

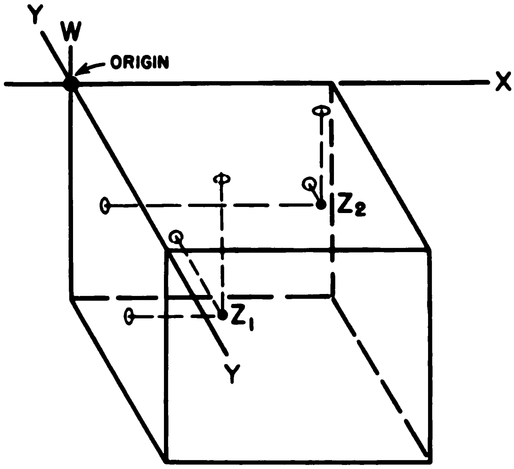

FIGURE 1.—Method of representing four variables in three-dimensional space. Three variables (*w*, *x*, and *y*) may be represented by values referred to three coordinate axes. Fourth variable (*z*) can be represented as series of values at specified points in three-dimensional space.

Suppose that we are faced with the problem of representing porosity trends in this block of rock. If the porosity varies in a regular manner, it might be represented by a surface. However, an ordinary surface embedded in three-dimensional space is inadequate because four variables are involved. Consequently we need a four-dimensional surface. A surface of four or more dimensions may be termed a *hypersurface*, the prefix "hyper" pertaining to above or beyond. Thus, a hypersurface is "above" or "beyond" an ordinary surface in a mathematical sense.

TRENDS AND THE LEAST-SQUARES CRITERION

In dealing with data that are irregular ("noisy"), we are commonly faced with the problem of establishing trends. For example, if observations of two variables are plotted on a two-dimensional diagram as a series of points (Fig. 2), the general trend of the points may be represented by a line. The trend line may be fitted by eye, but this is not particularly objective because one person might place the line differently than the next person.

The problem is to obtain the best fit of the line to the points. The most generally used criterion of best fit is that of *least squares*. In

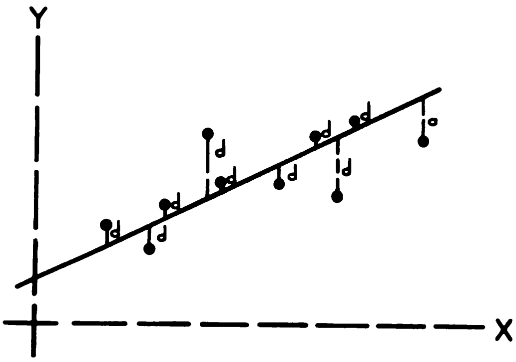

FIGURE 2.—Least-squares fit of line to points. Line has been fitted so that sum of squared deviations (marked with *d's*) of *y* with respect to *x* is minimized.

fitting a line by least squares, the objective is to fit the line so that the sum of the squared deviations of one variable, with respect to the other, is the least possible (Fig. 2). Thus, a least-squares fit is unique because only one position of a line will yield the least possible sum of squared deviations. However, it should be borne in mind that it makes a difference which variable is being minimized. In Figure 2, the trend line has been drawn so that the deviations of y with respect to x have been minimized. The line would have been fitted slightly differently had the objective been to minimize the deviations of x with respect to y. The reason for the difference is that we are not dealing with an ordinary functional relationship in which it makes little difference whether we express y as a function of x, or *vice versa*. Instead, we are dealing with a correlation in which we seek the best estimate of one variable in terms of the other, either y with respect to x, or x with respect to y.

The least-squares criterion is not confined to the fitting of straight lines. Curved lines described by mathematical functions can also be fitted by least squares. Furthermore, the least-squares criterion is applicable to the fitting of planes (Fig. 3), curved surfaces embedded in three-dimensional space, and hypersurfaces.

EQUATIONS AND THE LEAST-SQUARES CRITERION

Lines, surfaces, or hypersurfaces that have been fitted by least squares may be described by equations. For example, the equation describing a straight line may be generally written

$$y = A + Bx$$

where x and y are variables, and A and B are constants. In this equation, y is the dependent variable, x is the independent variable, A is the intercept value of the line on the y axis, and the coefficient, B, represents the slope of the line. It is understood that the algebraic sign, plus or minus, is incorporated within these constants. In fitting a straight line by least squares, the problem is to calculate the values of A and B so that the sum of the squared deviations is the least possible. In fitting curved lines, surfaces, or hypersurfaces by least-squares methods, the objective is the same, namely, to obtain the constants of the equations so that the sum of squared deviations is minimized.

The degree of an equation containing a dependent variable and one independent variable is related to the maximum values of the exponents. For example, a second degree equation may be written

$$y = A + Bx + Cx^2$$

in which x and y are variables and A, B, and C are constants. Similarly, a general equation of the third degree involving one dependent and one independent variable may be written

$$y = A + Bx + Cx^2 + Dx^3.$$

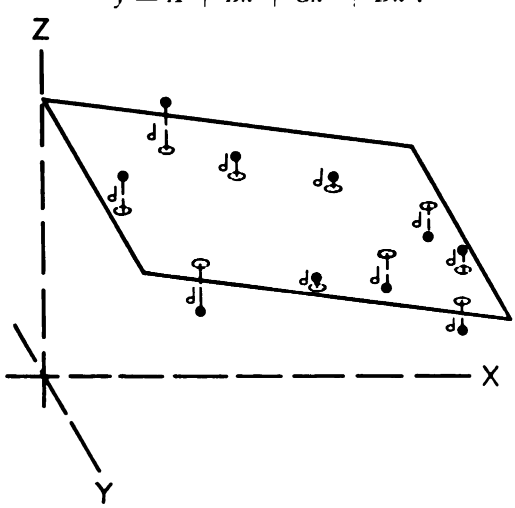

FIGURE 3.—Least-squares fit of plane to points.

At this point it is convenient to introduce a general classification of equations and their graphic representations according to degree and number of variables. Figure 4 presents a series of equations and their graphs in which degree is listed by column and number of variables by row. For example, in equations of the first degree, two variables yield a straight line, three variables a plane, and four variables a first-degree hypersurface.

The terms within each equation of Figure 4 may be classed according to whether they are linear, quadratic, or cubic. The linear components are those of the first degree, and include the intercept, *A*, and terms to the first power. The quadratic components include

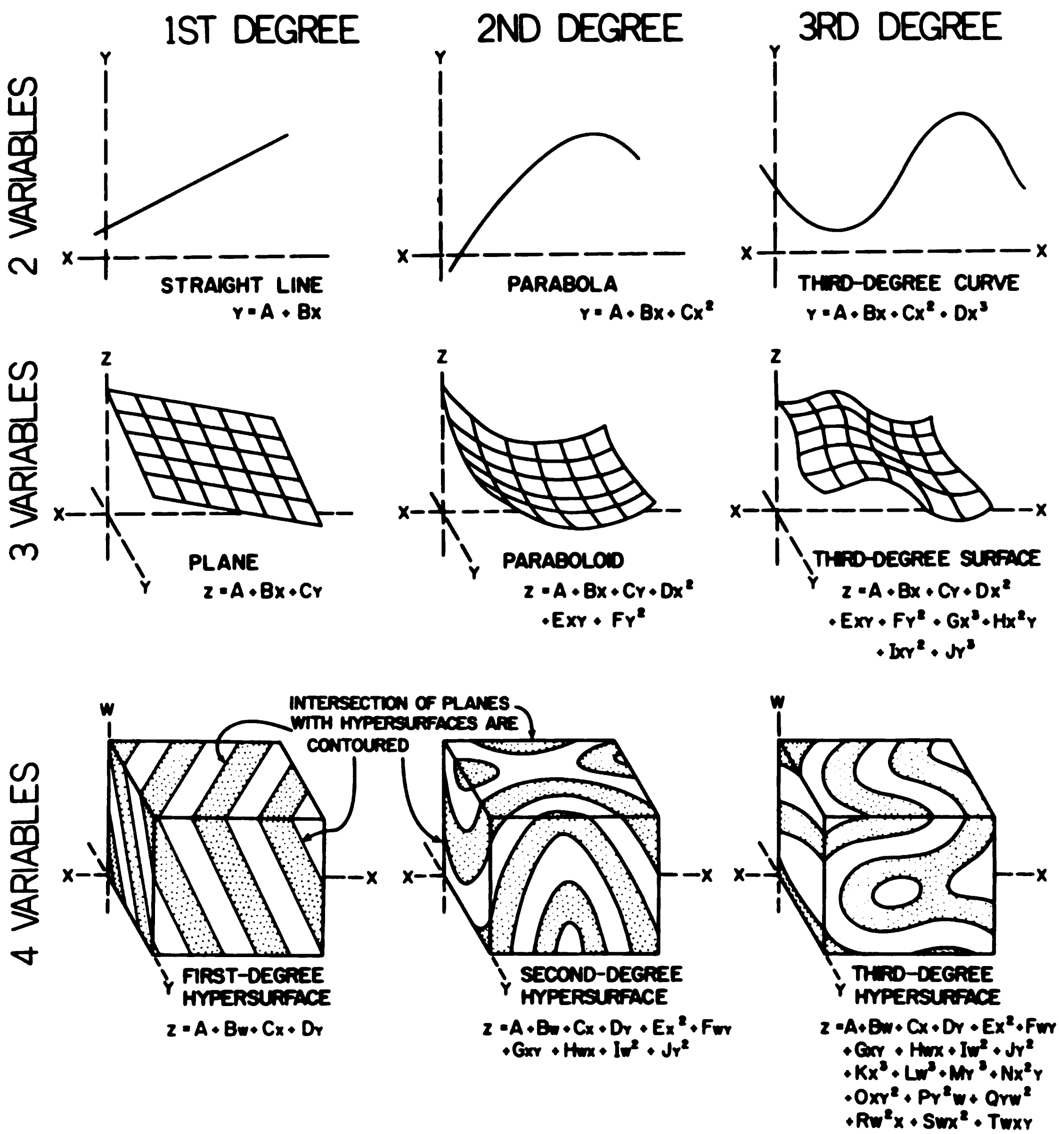

FIGURE 4.—Relationship between number of variables and degree of generalized equations and their geometric equivalents. Degree (first, second, and third) is listed by column and number of variables (two, three, or four) by rows. Variables are denoted by *w*, *x*, *y*, and *z*, and constants (with algebraic sign implicitly included) by *A* through *T*. Two variables are represented geometrically by straight or curved lines, three variables by surfaces, and four variables by hypersurfaces.

terms containing up to two variables to the first power, or one variable to the second power. Thus, in the equation of the three-variable second-degree surface of the general form (Fig. 4), the linear terms are $A + Bx$, and the quadratic terms are $Dx^2 + Exy + Fy^2$. In Table 1 the terms of the general equations of Figure 4 are classified according to whether they are linear, quadratic, or cubic.

VISUALIZING FOUR-DIMENSIONAL SURFACES

Four-dimensional surfaces (hypersurfaces) may be visualized. Consider the way in which four variables (w, x, y and z) may be represented by a coordinate system in three-dimensional space (Fig. 1). Three of the variables (w, x and y) may be represented as dimensions with respect to three reference axes arranged perpendicular to one another. The fourth variable, z, may be represented at individual points in space. To these points in space we may fit, by least squares, a plane or curving hypersurface which represents the best estimate of z in terms of the other three variables, w, x and y. We may visualize such a four-dimensional hypersurface as a series of infinitesimally thin, three-dimensional surfaces nested together. If the hypersurface is intersected by planes, as for example on the top, bottom, and four sides of a block (Fig. 4), the intersections of the hypersurface with the planes of the block may be portrayed by contour lines drawn on the surfaces of the block. A first-degree hypersurface (Fig. 4) might be likened to a series of parallel planes or series of slices, each infinitesimally thin. Higher-degree hypersurfaces may be thought of as formed by an infinite number of nested, curving surfaces rather than planes.

TABLE 1.—Generalized equations classified according to degree and number of variables. The dependent variable has been omitted here.

Number of variables	Degree	Descriptive title	Classification of terms in equation		
			Linear	Quadratic	Cubic
2	First	Straight line	$A + Bx$		
	Second	Parabola	$A + Bx$	$+ Cx^2$	
	Third	Third-degree	$A + Bx$	$+ Cx^2$	$+ Dx^3$
3	First	Plane	$A + Bx + Cy$		
	Second	Elliptic paraboloid or hyperbolic paraboloid	$A + Bx + Cy$	$+ Dx^2 + Exy + Fy^2$	
	Third	Third-degree surface	$A + Bx + Cy$	$+ Dx^2 + Exy + Fy^2$	$+ Gx^3 + Hx^2y + Ixy^2 + Jy^3$
4	First	First-degree hypersurface	$A + Bw + Cx + Dy$		
	Second	Second-degree hypersurface	$A + Bw + Cx + Dy$	$+ Ex^2 + Fwy + Gxy + Hwx + Iw^2 + Jy^2$	
	Third	Third-degree hypersurface	$A + Bw + Cx + Dy$	$+ Ex^2 + Fwy + Gxy + Hwx + Iw^2 + Jy^2$	$+ Kx^3 + Lw^3 + My^3 + Nx^2y + Oxy^2 + Py^2w + Qyw^2 + Rw^2x + Swx^2 + Twxy$

NOTE: Cubic terms with coefficients N through T have been arbitrarily omitted in this study but are listed here for the sake of completeness.

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OIL-GRAVITY VARIATIONS

One of the purposes of this report is to illustrate the use of four-variable hypersurfaces with an example. The example chosen deals with variations in crude oil gravity in southeastern Kansas. Here the problem is to interpret the geologic significance of differences in oil gravity from place to place, and from zone to zone stratigraphically. As an introduction to the problem, the measure of oil gravity is discussed first, followed by a discussion of oil-gravity variations in other regions.

MEASUREMENT OF OIL GRAVITY

API[1] gravity is the most widely used measure of the properties of crude oil. API gravity is a function of the density per unit volume, and its relationship to specific gravity is shown by the following formula:

$$\text{Degrees API} = \frac{141.5}{\text{Sp. Gr. at } 60^\circ\text{F}} - 131.5$$

1. American Petroleum Institute standard.

It should be noted that API gravity increases when specific gravity decreases, and *vice versa*. Thus, an oil with a high API gravity has a lower specific gravity than an oil with a low API gravity.

Certain other properties of crude oil are generally related to API gravity, including viscosity (which increases with decreasing API gravity) and gross chemical composition. API gravity is a rough measure of the proportions of hydrogen and carbon in crude oil; oils of high API gravity (low specific gravity) are richer in hydrogen than those of lower API gravity. It is believed that the API gravity of an oil is related to the conditions under which the oil originated, including the character of the organic source materials from which it was derived, the chemical and mineralogical composition of the rocks in which it is contained, and the physical conditions, such as temperature and pressure, under which it has "matured" and been stored.

OIL-GRAVITY VARIATIONS IN OTHER REGIONS

Oil-gravity Variations in the Gulf Coast

There are a number of references in the literature to oil-gravity gradients of a regional nature, or changes in oil gravity that may be correlated with changes in depth of burial. For example, Barton (1937) presented convincing evidence that the specific gravities of crude oils in the Gulf Coast region generally decrease with depth. When a particular Cenozoic stratigraphic interval is traced downdip, specific gravity of the oil decreases and API gravity increases. Barton suggested that the general decrease in specific gravity with depth reflects the evolutionary processes by which crude oils that were originally napthenic have been gradually converted into paraffinic crudes as an effect of temperature, pressure, and time. He suggested that these changes may be analogous to those in the refining of crude oil, in which the high temperatures and high pressures that prevail for a very short time in the refinery are capable of bringing about drastic changes in the chemical composition of petroleum. Underground, the increases in temperature and pressure that have accompanied deeper burial are much less severe than those encountered in the refinery, but Barton pointed out that the far greater amount of time available geologically may have compensated for less severe temperature and pressure conditions.

Barton's views on Gulf Coast crude oils were challenged by Haeberle (1951) and Bornhauser (1950), who stated that while an increase in API gravity (decrease in specific gravity) can generally be correlated with an increase in depth, the increase in API gravity is not necessarily a simple function of depth of burial, but instead, could be a result of facies changes. As the Cenozoic strata of the Gulf Coastal Region of Texas and Louisiana are traced downdip, they generally exhibit a progressive change from continental facies, to shallow-water marine, and, finally, to deep-water marine facies. In other words, if one wished to ignore changes in depth of burial entirely, one could make an almost equally strong case for control of oil gravities by facies alone. Thus, the deep-water marine sediments, consisting mostly of shale, yield oil of highest API gravity, whereas near-shore sediments, which contain larger proportions of sand, yield oil that is lower in API gravity. Due to the imbricate, wedge-like aspect of the strata of the Gulf Coast, a well tends to pass downward from near-shore sediments to deeper-water sediments. Thus, the oil-gravity changes encountered in different reservoirs in a single well, or the changes of oil gravity in a series of wells in which a given stratigraphic horizon is followed downdip, both tend to exhibit changes in oil gravity that could be interpreted as facies-controlled or depth-controlled. Obviously, in the Gulf Coast we are dealing with a problem in which correlations are simple enough to establish, but cause and effect relationships are more obscure.

Oil-gravity Variations in Wyoming

Hunt (1953) studied the variations of API gravity in crude oils in Wyoming, where the geology is more complicated than in the Gulf Coast. He came to two principal conclusions: (1) There is a strong correlation of API gravity and other measures of the composition of crude oils with environment of deposition of the reservoir rocks in which the oils occur. Relatively low API gravity oils are associated with Paleozoic sediments, formed under quiet, stable conditions of moderate to high salinity, in which carbonates and sulfates were abundant. High API gravity oils tend to be associated with Mesozoic sediments, formed under conditions of moderate tectonic activity, in which dark shales predominate, with discontinuous sandstones and a few thin beds of limestone. Thus, environment of deposition, including the character of organic source materials, seems to be the most important factor affecting API gravity in Wyoming. (2) There is, however, a relationship between depth of burial and API gravity in Wyoming, provided that the oils are separated into two major groups, Paleozoic and Mesozoic. Hunt found that there is an overall increase in API gravity with depth of occurrence of oils in Paleozoic rocks and similarly with oils in Mesozoic rocks. However, the deepest Paleozoic oils are of a lower API gravity than are the shallowest Mesozoic oils. Hunt's (1953, p. 1865) plot of API gravity versus depth of oil in the Paleozoic Tensleep Formation suggests that there is an almost linear increase in API gravity with depth. Hunt concluded that depth of burial cannot be ignored, but that it is of secondary importance.

Oil-gravity Variations in Western Canada

Hitchon *et al.* (1961) showed that there is an apparent progressive increase in API gravity downdip east of the Canadian Rockies in Mississippian, Pennsylvanian, Permian, Triassic, Jurassic, and some Cretaceous and Devonian strata. However, there is no regular increase in API gravity downdip in certain other Devonian and Cretaceous strata in the region. The cause of geographic variations in API gravity in western Canada is poorly understood. Hitchon *et al.* (1961, p. 296) suggest that in some stratigraphic units, high API gravities tend to occur in tectonic basin areas and low API gravities in shelf areas.

Colorado Shale Oil-gravity Variations

Smith (1963) pointed out that the specific gravity of oil produced from the Green River oil shales in Colorado decreases systematically with increasing depth of burial. Smith fitted by least squares a series of second-degree (parabolic) curves relating specific gravity to depth in individual bore holes. He stated that the decrease in specific gravity is associated with a progressive decrease in oxygen content with depth, which in turn may have resulted from loss of carboxyl groups from organic molecules due to increase of heat and pressure with increasing depth.

PREVIOUS STUDY OF OIL-GRAVITY VARIATIONS IN SOUTHEASTERN KANSAS AND VICINITY

A research committee of the Tulsa Geological Society, consisting of Neumann and others (1947), conducted a study of variations in crude oil in southeastern Kansas and adjacent northeastern Oklahoma. They concluded that the environment of deposition and the original character of the oil's organic source material probably determined the kind of oil in each pool. For example, they found that the oil in the "Bartlesville sand" of Osage County, Oklahoma, could be divided into six classes on the basis of distillation fractions. Pools in the "Bartlesville sand" containing particular classes of oil have distinct geographical groupings. Neumann's committee suggested that the area in which a particular class of oil occurs reflects a particular set of depositional conditions which prevailed in that area. They found little evidence that the oil migrated over appreciable distances, and they concluded that the oil formed mostly from organic materials deposited close to the places where the oil now occurs.

Recent findings by Baker (1962) support the conclusions of Neumann's committee. Baker compared the distribution of traces of hydocarbons in non-reservoir facies close to the shoestring sand reservoirs in the Pennsylvanian Cherokee Group ("Bartlesville sand" or "Burbank sand"), in the Thrall (Thrall-Aagard) field in Greenwood County, Kansas, and in the Burbank field in Osage County, Oklahoma. Baker found that the proportions of hydrocarbons (expressed as the ratio of saturate hydrocarbons to aromatic hydrocarbons) in the non-reservoir facies tend to parallel those of the crude oil produced in the adjacent oil fields. He found that traces of hydrocarbons extracted from the non-reservoir facies encountered in a core in the Burbank field have significantly higher saturate to aromatic ratios than hydrocarbons from non-reservoir facies close to the Thrall field. Burbank crude also has a higher saturate to aromatic ratio than Thrall crude. It is presumed that differences in the crude oils reflect differences in trace hydrocarbons extracted from associated, non-reservoir rocks. Consequently, both trace hydrocarbons and crude oil appear to have a similar source within a given locality.

OIL-GRAVITY DATA IN SOUTHEASTERN KANSAS USED IN THIS STUDY

Oil-gravity data used in this study were taken from a report by Everett and Weinaug (1955) and include API gravity measured at 60° F, well location, depth to producing zone, and name of producing zone. The oil-gravity data were studied in a rectangular area (Fig. 5) about 65 by 70 miles in dimension, which embraces Chautauqua, Cowley, and Elk counties, and parts of Greenwood, Butler, Woodson, Wilson, and Montgomery counties. The location of wells for which API gravity was determined is shown in Figure 5 and the wells are numbered by Everett and Weinaug (1955, p. 211-221) as follows: 8, 13 to 22, 30 to 35, 41 to 44, 55 to 93, 95 to 104, 107 to 137, 145 to 222, 224, 227 to 230, 232, 234 to 250, 386 to 397, 400, 404 to 406, 411 to 419, 421 to 426, 428, 444 to 447, 449 to 453. Data on wells listed by Everett and Weinaug that lie outside the area of this study were not used. A total of 244 API gravity values were used. The geographic distribution of wells yielding oil-gravity data is somewhat uneven, due largely to the uneven distribution of oil fields within the area (Fig. 6). In addition, the distribution of the gravity values according to well depth is also somewhat uneven. Accordingly, the data points used in this study are not randomly distributed in space.

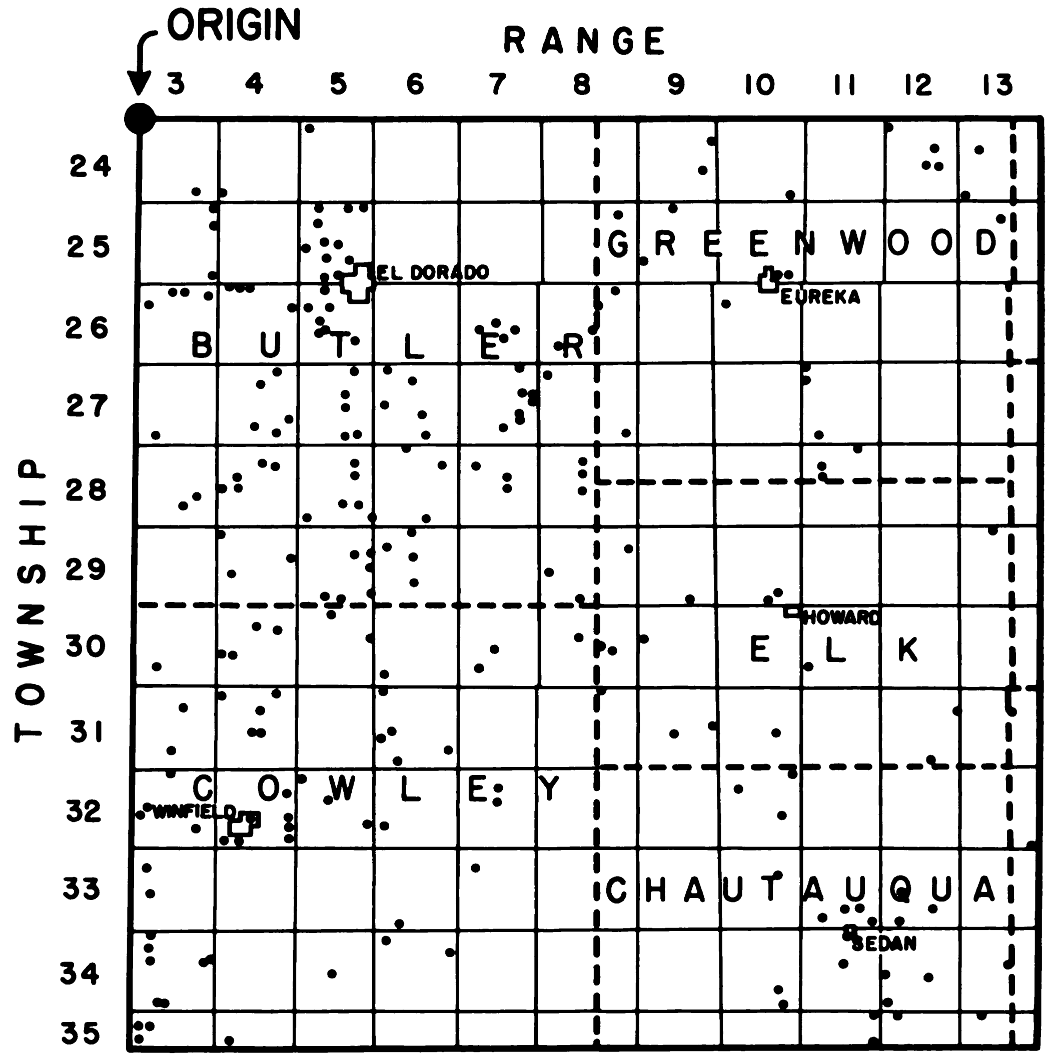

FIGURE 5.—Map of part of southeastern Kansas showing location of oil wells yielding oil-gravity data used in this study.

Classification of Oil-Producing Zones Yielding Oil-Gravity Data

Oil is produced from various stratigraphic zones in the area of this study in southeastern Kansas (Table 2). The names of some of the zones are local drillers' terms that are not official geological names. Jewett (1954, p. 76-90) provides a glossary of names of oil-producing zones in eastern Kansas, and the approximate stratigraphic position of the zones is given in a columnar chart by Jewett (1959).

Distribution of Oil Gravities on a Zone-by-Zone Basis in Southeastern Kansas

The distribution of API gravities in six stratigraphic zones in part of southeastern Kansas is shown on maps in Figure 7. The stratigraphic position of each zone is given in Table 2. The maps show that (1) in detail, areal variations in API gravities are erratic, but (2) that broad scale trends are present. API gravities of oils in the Arbuckle Limestone and Kansas City Group (Fig. 7*A*, 7*E*)

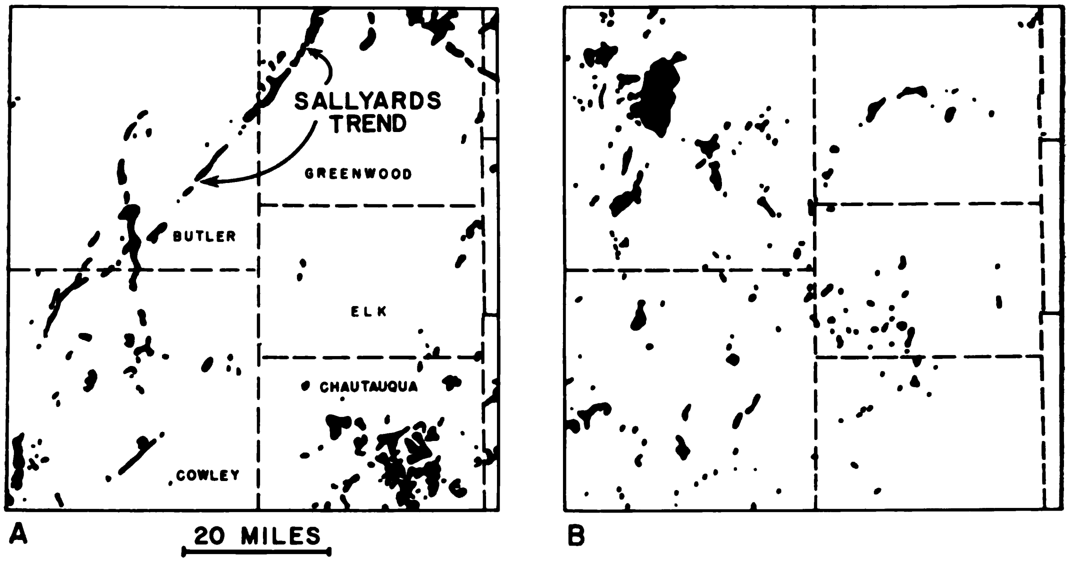

FIGURE 6.—Maps showing outlines of oil fields in part of southeastern Kansas (modified from Goebel, Hilpman, Beene, and Noever, Pl. 1, 1962). (*A*) Oil fields in which oil occurs principally in lenticular sands or in shoestring sands, and in which accumulation of oil is mainly stratigraphically controlled. (*B*) Oil fields in which oil occurs principally in carbonate reservoir rocks and in which structural control of oil accumulation is important.

generally increase toward the west, and API gravities in the "Mississippi lime" and "Mississippi chat" and "Layton sand," (Fig. 7*B*, 7*D*, 7*F*) generally increase toward the northwest. API gravities in the "Bartlesville sand" (Fig. 7*C*) are more erratic, and gross changes across the area are not apparent.

Considering the oil-producing zones in general, there is a down-dip increase in API gravity, the regional structure being a west-dipping homocline (Fig. 8).

Thus, the question arises, is the increase in API gravity toward the west due to increasing depth, or is it related to geographic position?

Some other aspects of the geographic distribution of API gravities are worth noting. The distribution of API gravities in the "Bartlesville sand" (Fig. 7*C*) appears to parallel the "Sallyards shoestring" trend (Fig. 6*A*). Bass *et al.* (1937) have interpreted this trend, as well as other shoestring sand deposits, to be ancient offshore bars formed at the shifting margin of a Pennsylvanian sea. Perhaps variations in environmental conditions during Pennsylvanian time are responsible for much of the variation in oil gravities in the "Bartlesville sand."

Within the area, the "Bartlesville sand" has the highest average API gravity, with values ranging from a little less than 36° API to greater than 40° API (Fig. 7*C*). However, the range of gravity variations is greater in oil obtained from the "Mississippi lime" and the "Mississippi chat" (Fig. 7*B*). Some geologists have speculated that oil in the "Mississippi chat" and "Mississippi lime" has been derived from shale in the overlying Cherokee Group, which contains the "Bartlesville sand" and other oil-producing sands. However, the contrast of API gravities in the "Mississippi lime" and "chat" with those in the "Bartlesville" suggests that the oils may be of differing sources.

HYPERSURFACES FITTED TO OIL-GRAVITY DATA IN SOUTHEASTERN KANSAS

First-, second-, and abbreviated third-degree trend hypersurfaces have been fitted to oil-gravity data in southeastern Kansas, and the results are appraised statistically and geologically below. A glossary of statistical terms

TABLE 2.—Local terms and stratigraphic position of oil-producing zones in area of study.

Oil-producing zone	Group	Stage	System
Admire	Admire	Gearyan	Permian
Topeka Limestone	Shawnee		
"Peacock sand"			
"Hoover sand"		Virgilian	
"Stalnaker sand"	Douglas		
Lansing	Lansing		
"Layton sand"		Missourian	
"Kansas City lime"	Kansas City		Pennsylvanian
"Wayside sand"	Marmaton		
"Peru sand"			
"Cattleman sand"		Desmoinesian	
"Bartlesville sand"*	Cherokee		
"Burgess sand"			
"Mississippi chat"			Mississippian
"Mississippi lime"		Meramecian	
Viola Limestone		Middle Ordovician	
"Simpson sand"	Simpson		Ordovician
Arbuckle Limestone	Arbuckle	Lower Ordovician	

* NOTE: "Bartlesville sand" is a general name given certain lenticular oil-producing sands that vary slightly in age and stratigraphic position from place to place.

used but not explained otherwise is provided later in this report.

STATISTICAL APPRAISAL

Frequency Distribution of Values

Frequency distribution of original oil-gravity data and of first-, second-, and third-degree trend and residual (deviation) values is shown in a series of histograms in Figure 9. The original data (Fig. 9*A*) are somewhat skewed so that the mean is displaced to the left, or low side, of the median. The distribution of natural logarithms of the original data (Fig. 9*B*) is more symmetrical, although logarithmic transformations of original data were not deemed necessary in this study.

The characteristics of the frequency distributions of trend and residual values are important in analysis of variance to determine confidence levels because more or less symmetric frequency distributions of trend and residual values are desirable. The histograms (Fig. 9) of residual values (Fig. 9*C*, *E*, *G*) reveal moderate skewness, partly reflecting the skewness of the original data. The distributions of the trend values (Fig. 9*D*, *F*, *H*) are also somewhat skewed. However, it is concluded that the frequency distributions are not sufficiently skewed to invalidate use of analysis of variance.

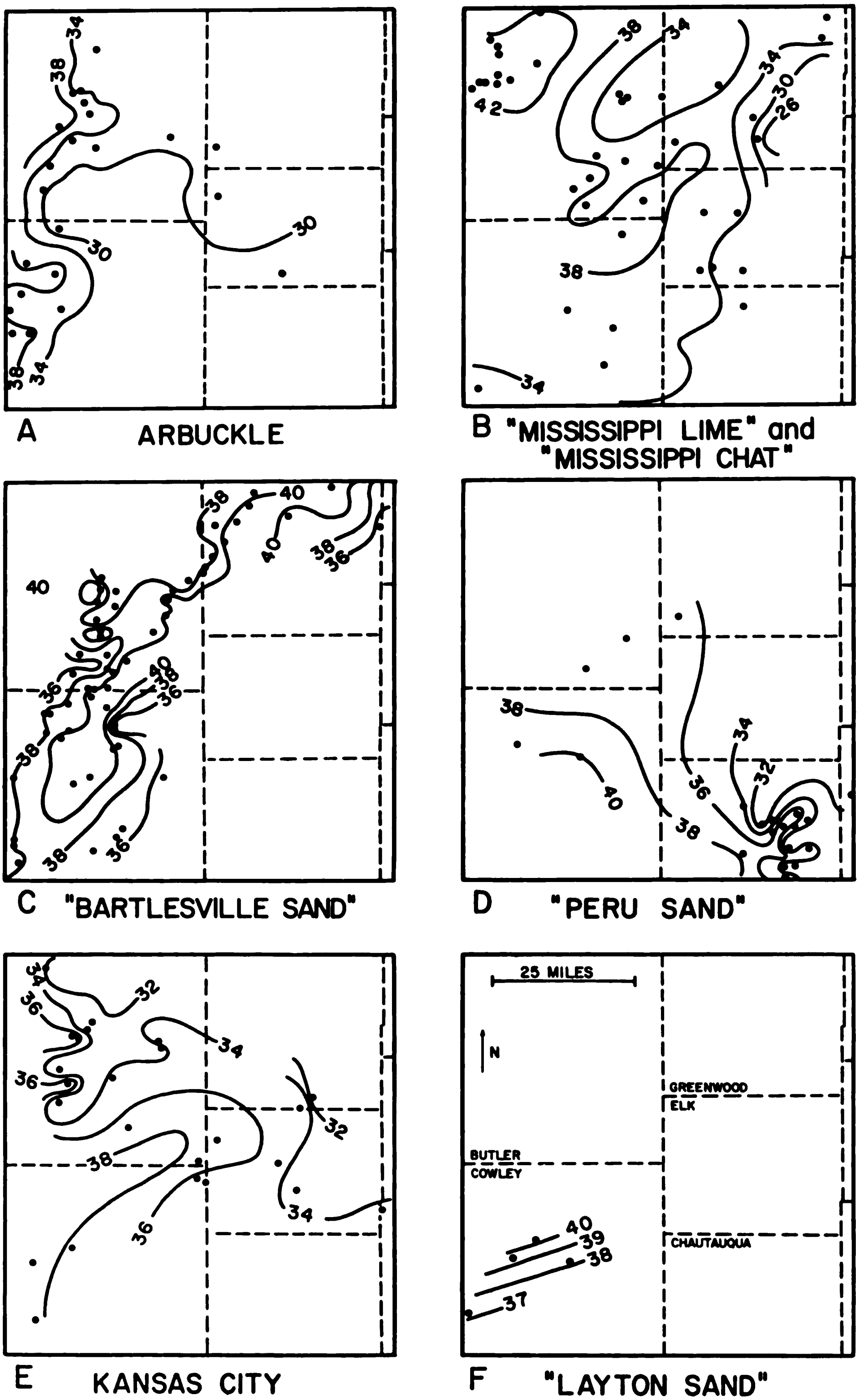

FIGURE 7.—Contour maps showing variations of API gravity in different stratigraphic zones in southeastern Kansas. (Dots mark locations of oil wells.) (See Fig. 5 for location of township and range.)

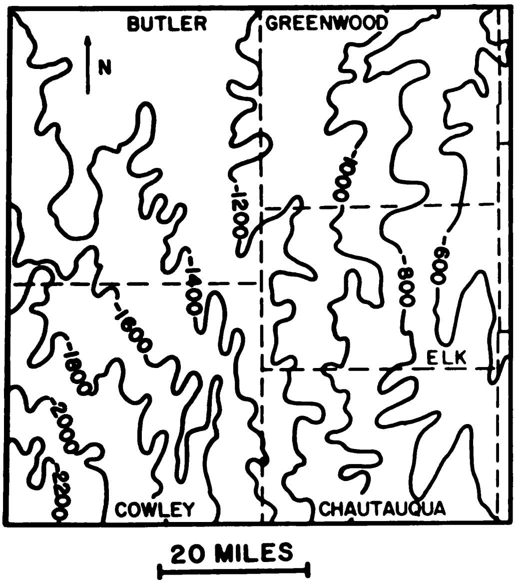

FIGURE 8.—Structure contour map on top of Mississippian rocks in part of southeastern Kansas (after Merriam, 1960). Contour values in feet. (See Fig. 5 for location of township and range.)

Confidence Levels of Trend Components

Analysis of variance may be used to determine the statistical significance of trend surfaces (Dawson and Whitten, 1962, p. 8; and Allen and Krumbein, 1962, p. 522-523). In this study, the objective has been to determine the degree of confidence for each component of the hypersurfaces, or, in other words, to determine whether the linear, quadratic, and cubic components are statistically significant or could be due to chance alone. The degree of confidence is spoken of as the "confidence level," and may be expressed in percent. On this basis, absolute certainty is 100 percent, and absolute uncertainty is 0 percent. A confidence level of 99 percent for a particular component would indicate 99 percent certainty that the component represents a real effect and not chance.

Table 3 includes the basic data for calculation of confidence levels by analysis of variance. The data include (*a*) sum of squares

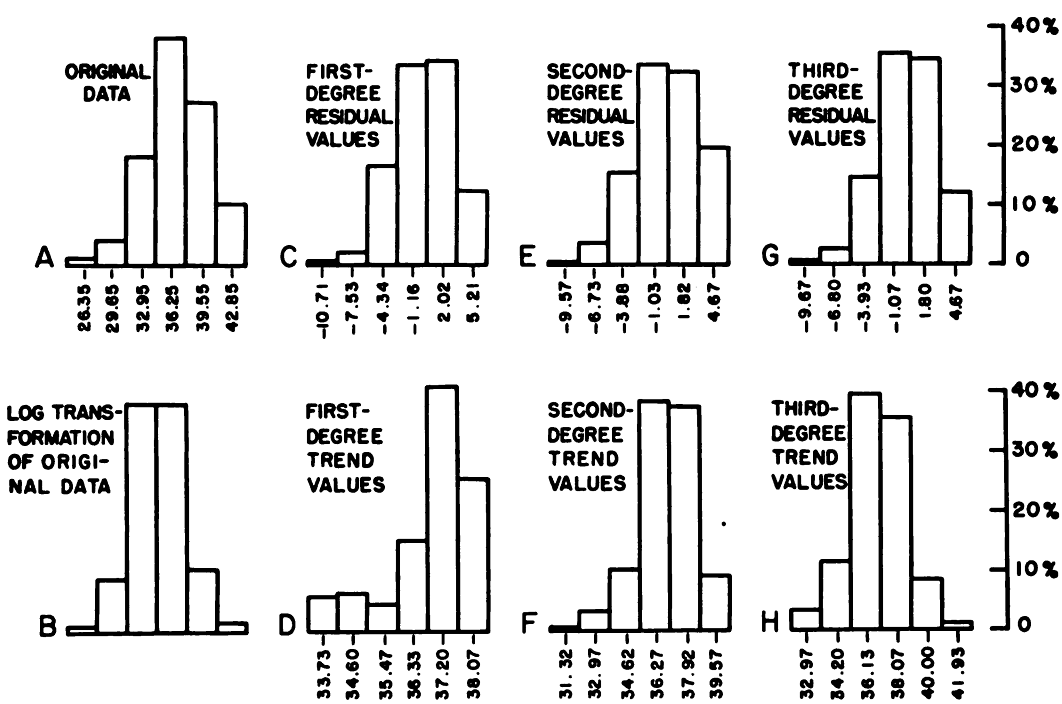

FIGURE 9.—Histograms of frequency distributions of original API gravity data, logarithmically transformed data, and trend and residual values. Numbers beneath histograms refer to midpoint values within each frequency class.

TABLE 3.—Analysis of variance of oil-gravity trend hypersurface data.

Source	Sum of squares	Degrees of freedom	Mean square	Snedecor's F	Confidence level
Total, 244 data points	333,019.5	243	....	....	....
Due to linear component	330,624.2	3	110,208.1	10,054.0	99.9+%
Deviations from linear	2,395.3	240	10.0	....	....
Due to quadratic component	252.6	6	42.1	4.6	99.9+%
Deviations from quadratic	2,142.7	234	9.1	....	....
Due to abbreviated cubic component	189.1	3	63.0	7.5	99.9+%
Deviations from abbreviated cubic	1,953.6	231	8.5	....	....

that are apportioned among the linear, quadratic, and abbreviated cubic components, respectively, (*b*) sums of squares associated with the deviations or residuals, and (*c*) number of degrees of freedom associated with the components and the deviations. These data, in turn, permit (*d*) calculation of the mean square of the components and deviations and (*e*) calculation of Snedecor's *F*. Finally, (*f*) the confidence level in percent is obtained by reference to tables of *F* (Snedecor, 1956, p. 246-249).

The number of degrees of freedom is established in reference to (1) the number of degrees of freedom associated with the total number of data points (n — 1), and (2) number of terms containing variables in the equation belonging to each component. Thus, there are three degrees of freedom associated with the three linear terms Bw, Cx and Dy, six degrees of freedom with the six quadratic terms Ex^2, Fwy, Gxy, Hws, Iw^2, *and* Jy^2, and three with the cubic terms Kx^3, Lw^3, My^3. The number of degrees of freedom at each level is obtained by successively subtracting the degrees of freedom associated with each component from the degrees of freedom associated with the data points.

The confidence levels associated with the three trend components of the oil-gravity data are extremely high, all being in excess of 99.9 percent. It is concluded that the effect associated with each component is real and not fortuitous.

Percent Total Sum of Squares Represented by Hypersurfaces

The percent of total sum of squares is a measure of how closely the hypersurfaces (Table 4) fit the observed data and is calculated according to an equation given in Appendix *A*. A percent of total sum of squares of 100 percent would represent a perfect fit of the observed data. There is a general relationship between the confidence level associated with a component, and the percent of total sum of squares associated with that component.

TABLE 4.—Percent of total sum of squares represented by hypersurfaces fitted to oil-gravity data.

Linear surface	32.0%
Linear + Quadratic surface	49.7%
Linear + Quadratic + Abbreviated Cubic surface	63.0%

If there is a marked increase in percent of total sum of squares when a new component is included, a high confidence level is generally associated with that component (Table 3) and *vice versa*.

Calculation of Weighted Averages of Oil Gravity

Spatially weighted averages (Table 5) of API gravity have been calculated for each hypersurface by the method described in Appendix *A*. The resulting averages are close to the arithmetic mean. Little advantage is gained in this case by calculation of spatially weighted averages.

INTERPRETATION OF TREND HYPERSURFACES

Trend hypersurfaces fitted to API gravity data are shown in block diagrams (Fig. 10) in which contour lines portray the intersections of sides of the blocks with the hypersurfaces.

TABLE 5.—Averages (API degrees) of oil-gravity values in southeastern Kansas.

Arithmetic mean	36.79°
Average value within first-degree hypersurface	35.87°
Average value within second-degree hypersurface	36.33°
Average value within third-degree hypersurface	36.96°

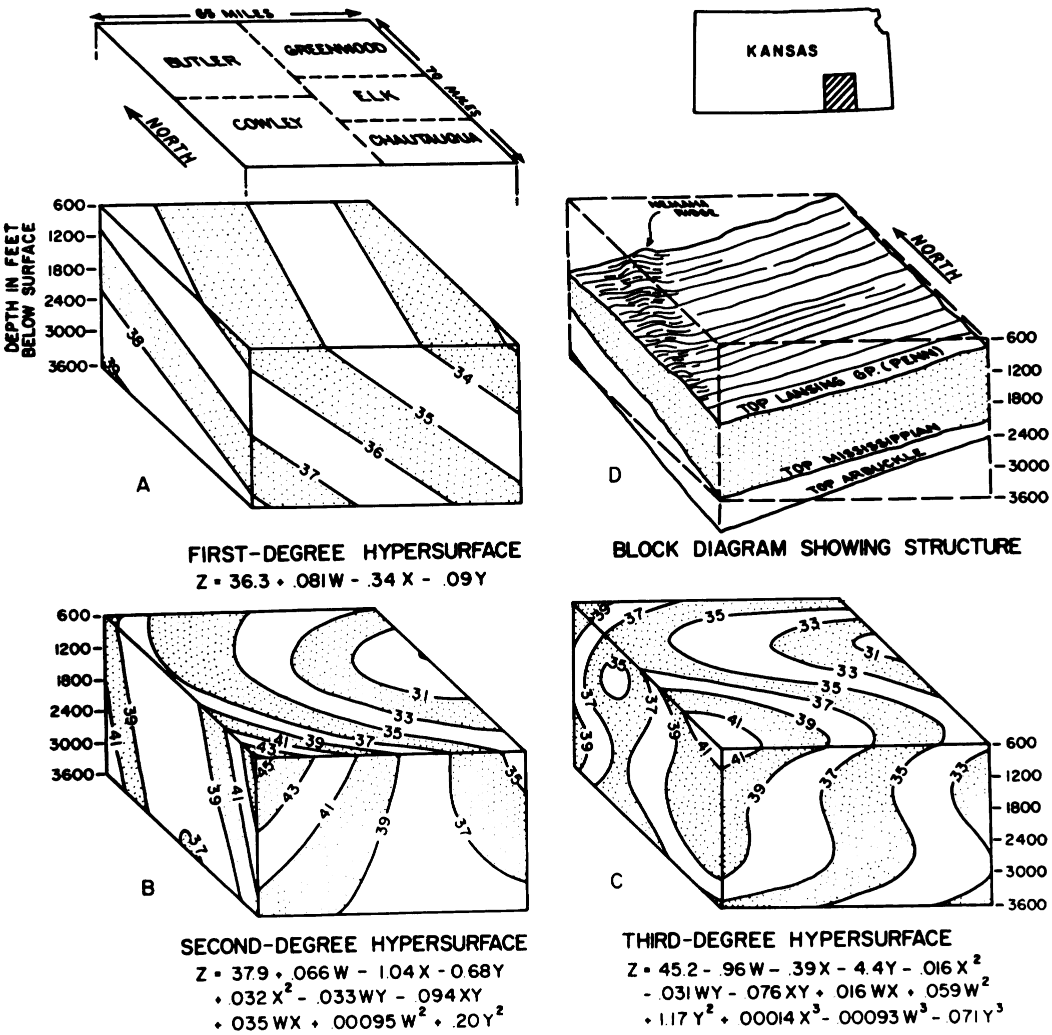

FIGURE 10.—Hypersurfaces (*A-C*) fitted to API gravity data with respect to depth below surface and geographic location. Block *D* shows generalized geologic structure. Equations of hypersurfaces are listed.

First-degree Hypersurface

The first-degree hypersurface (Fig. 10*A*) represents the observed data moderately well, accounting for about 32 percent of the total sum of the squares (Table 4). The confidence level (Table 3) is in excess of 99.9 percent, signifying that the first-degree trend hypersurface represents a real effect and cannot be due to chance alone.

The first-degree hypersurface may be likened to an east-southeast-dipping homocline, reflecting trends in the original data, namely that (1) at a given depth there is a general increase in API gravity toward the west-northwest, and (2) at any particular locality, there is a general increase in API gravity with depth. The first-degree hypersurface makes clear that differences in API gravity are not segregated in any uniform manner according to depth or to stratigraphic zones because the planes within the first-degree hypersurface dip toward the east, whereas the strata dip generally toward the west (Fig. 8; 10*D*).

Second-degree Hypersurface

The second-degree hypersurface (Fig. 10*B*) reveals trends that differ considerably from the first-degree hypersurface. The second-degree hypersurface might be likened to a complex syncline that plunges toward the east-southeast on one side, but the direction of plunge and shape of the hypersurface are gradually reversed, as is revealed by contours on the south-facing or front side of the block. We are dealing with a series of complex, nested surfaces within the hypersurface, and the shape of any particular surface, as for example the 37° API surface, is that of a saddle-shaped hyperbolic paraboloid. The second-degree hypersurface represents a percent of the total sum of the squares of about 50 percent (Table 4), and the confidence level associated with the quadratic component is in excess of 99.9 percent (Table 3).

Interpretation of the geologic significance of the second-degree hypersurface is somewhat difficult because the surface is more complex than the first-degree hypersurface. The increased complexity reflects the improved fit of the hypersurface and emphasizes that oil-gravity values vary in a complex manner within the area.

Third-degree Hypersurface

Although the third-degree hypersurface (Fig. 10*C*) is still more complex than the second-degree hypersurface, there are marked similarities between the two. Surfaces within the third-degree hypersurface may be likened to an eastward-plunging, complex syncline that gradually becomes a saddle-shaped structure. The percent of total sum of squares represented by the third-degree surface is about 63 percent (Table 4), and the contribution of the abbreviated cubic component (the cubic cross product terms have been omitted) is real, since a confidence level of more than 99.9 percent is associated with it (Table 3).

The third-degree hypersurface also reflects the increase in API gravities toward the west. but it suggests that the increase is by no means a simple increase in that direction. Furthermore, it appears to bear a relationship to the low API gravities in the Arbuckle, or lowest oil-producing zone, as indicated by the westward deflection of the contour lines near the bottom of the south-facing side of the block (Fig. 10*C*).

Spatial Distribution of Residual Values

The generalized spatial distribution of positive and negative second-degree residual oil-gravity values is shown in a series of "slice maps" (Fig. 11*A*), and distribution of positive residuals is shown in a block diagram (Fig. 11*B*). The residual values were obtained by subtracting trend values from observed values. The spatial distributions of first- and third-degree residual values are almost the same as the second-degree residuals and are not shown here. Considering the positive residuals within the block extending from 1200 to 3600 feet (well depth), there are three main places where positive values congregate: (1) in the extreme southeast corner of the block, (2) in the extreme northwest corner of the block, and (3) in a broad and very irregular zone that extends from northeast to southwest across the block. The negative residuals are aggregated between the positive residuals.

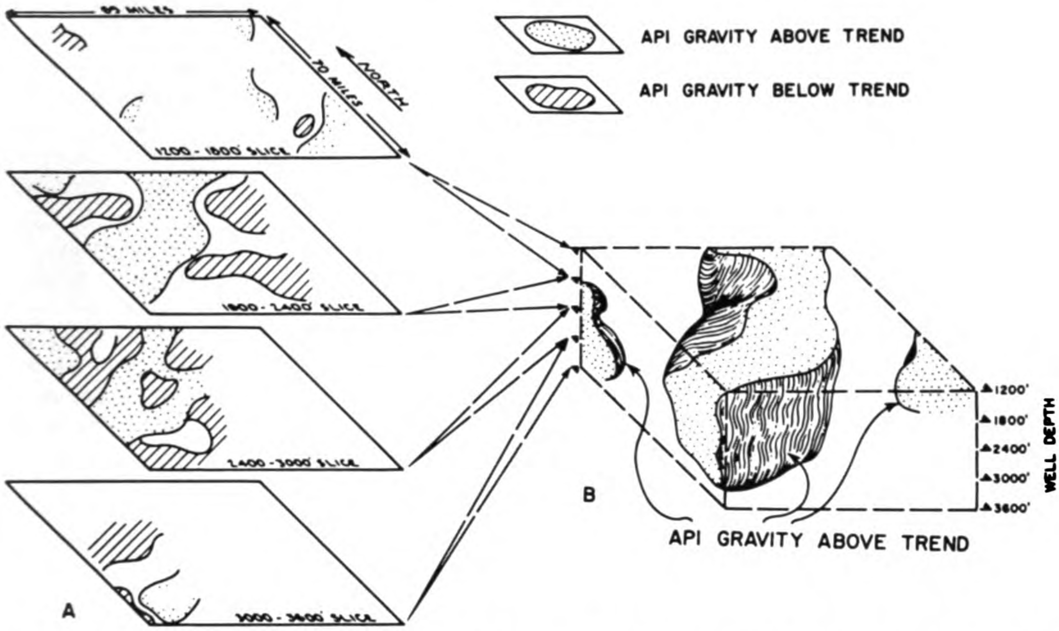

FIGURE 11.—Distribution of second-degree residual values of API gravity in three-dimensional space. (*A*) Series of four slice maps, each representing 600-foot depth interval, showing clustering of residual values. (*B*) Block diagram showing clustering of positive residual values.

The clusters of residual values reflect variations in the original data. The positive cluster that trends northeast-southwest across the block partly reflects high API values in the "Bartlesville sand" (Fig. 7*C*) in the Sallyards trend (Fig. 6*A*). However, it is interesting to note that this cluster also includes oil-producing zones that lie both above and below the "Bartlesville sand." This is important because it suggests that factors responsible for relatively high API gravities in this cluster are common to various stratigraphic zones and not just the "Bartlesville" alone.

An interesting comparison can be made between the areal distribution of structural residual values and API oil-gravity residual values. Figure 12 shows the clusters of structural residual highs and lows produced when a second-degree, three-variable trend surface is subtracted from the structure on top of Mississippian rocks in this same part of southeastern Kansas (Fig. 8). There are several structural residual lows which trend in a northeast-southwest direction across the area. This northeast-southwest trend coincides more or less with the cluster of positive oil-gravity residual values (Fig. 11). In addition, there are residual structural lows toward the southeast and northwest corners of the area, roughly in the same places where the other two API gravity residual highs occur. Structural highs of the Nemaha ridge (Fig. 8; 10*D*) seem associated with oil-gravity lows, but this may be due in presence of relatively low-gravity Arbuckle fields (Fig. 6*B*; 7*A*) in that part of

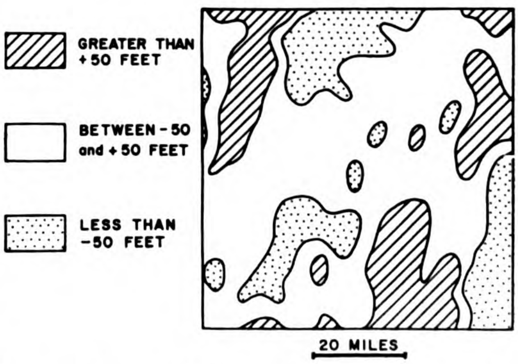

FIGURE 12.—Simplified contour map of residual values obtained by subtracting second-degree trend surface fitted to structure on top of Mississippian rocks (Fig. 8) in area of study.

the area. The coincidence between residual oil-gravity highs and residual structural lows may not be fortuitous, and perhaps similar relationships occur in other areas.

The tendency for oil-gravity residual values from different stratigraphic zones to be aggregated into the same clusters may reflect long-persisting ancient geographic and environmental conditions. For example, during part of Pennsylvanian time, limestone marine banks that formed in southeastern Kansas (Harbaugh, 1960, p. 229-232) tended to be stacked one upon another in the same general localites, in spite of occurring in different stratigraphic units. It is presumed that the marine banks were a localized response to environmental conditions that included water depths, waves, currents, and sources of terrestrially derived sediment. These environmental conditions, in turn, probably reflected large-scale ancient geographic conditions, such as the configuration of land and sea. It is speculated that oil-gravity residual clusters in southeastern Kansas may also partly reflect ancient geographic and environmental features that may have persisted during much of the Paleozoic Era. For example, the ancient geography may have influenced the distribution of marine organisms, including the phytoplankton, and in turn, influenced the characteristics of crude oil formed from organic material incorporated into the sediments. However, these suggestions are tentative and additional study is needed before final conclusions are drawn.

SUMMARY OF INTERPRETATIONS OF OIL-GRAVITY VARIATIONS IN SOUTHEASTERN KANSAS

1. There is broad correlation between well depth and API gravity: API gravities tend to increase with depth.

2. Factors other than depth appear to have a strong influence, however. These factors might generally be classed as depositional environment factors.

3. The first-degree hypersurface (Fig. 10*A*) makes clear that oil-gravity variations are not controlled by progressive changes between stratigraphic zones, because planes within the first-degree hypersurface dip opposite to the strata (Fig. 10*D*).

4. Similar environmental factors may have influenced oil gravities in different stratigraphic zones in the same general localities. This is suggested by aggregation of residual values in distinct clusters in three-dimensional space.

5. The tendency for residual values to be clustered suggests that depositional conditions affecting oil gravities in a given locality may have remained more or less the same during much of the Paleozoic. If this is the case, residual clusters could represent responses to long-persisting ancient geographic features, such as shore lines, sediment source areas, and organism communities, which affected the depositional environment at any particular place for long intervals of time.

6. It is suggested that both depth of burial and depositional environment have influenced oil-gravity values. Of the two, perhaps depositional environmental factors are the most important.

GLOSSARY OF STATISTICAL TERMS USED BUT NOT EXPLAINED ELSEWHERE IN THIS REPORT

Analysis of variance.—A technique in which the variation within a set of data is separated into different components, permitting differences between and within components to be compared. Ordinarily, the estimate of variance is:

$$V = \sigma^2 = \frac{\Sigma(x - \bar{x})^2}{n - 1},$$

where V = variance,
n = number of data values,
x = observed data values,
$\bar{x}$ = arithmetic mean,
σ = standard deviation.

However, in this study, analysis of variance was used to determine the significance of trend hypersurfaces, and mean square values were used instead of variance estimates.

Degrees of freedom.—Pertains to the number of opportunities in which variation may occur.

For example, a set of data containing ten data values has nine degrees of freedom. Similarly, a set with one data value contains zero degrees of freedom, because no variation is possible.

Snedecor's F.—The ratio of two variances, or the ratio of two mean squares.

Frequency distribution.—Pertains to the manner in which a set of data values are distributed according to frequency of occurrence.

Lorgarithmic transformation.—Involves use of logarithms of data values rather than the raw data values themselves.

Mean square.—Refers to the sum of squares divided by the number of degrees of freedom:

$$\text{mean square} = \frac{\text{sum of squares}}{\text{degrees of freedom}}.$$

Residual values (deviations).—Obtained by subtracting trend values from observed values.

Skewness.—Pertains to the degree of asymmetry in a frequency distribution.

Sum of squares.—The sum of squared values.

Trend values.—Values estimated on the basis of a trend line or surface. For example, if a trend line is fitted to points on an X-Y diagram, for each value of Y there is a corresponding estimate of X.

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APPENDIX *A*

DESCRIPTION OF COMPUTER PROGRAM FOR FITTING FOUR-VARIABLE HYPERSURFACES

General Statement

Details of the computer program used in fitting, contouring, and evaluating four-variable trend hypersurfaces are presented in Appendix *A*. Appendix *B* is a complete listing of the computer program in which cards or lines are identified by number, and Appendix *C* consists of reproductions of examples of output from the program.

This program is written in a computer language called BALGOL, which is one of several "dialects" of the computer language termed ALGOL-58. A language such as BALGOL is termed a "source language." The program is placed initially on punched cards and then read into the computer where it is translated into machine language which the computer can utilize directly. The translation is accomplished by using another program, termed a compiler, which is usually recorded on magnetic tape and which translates, or compiles, the source language.

The program described here has been written primarily for use on either an IBM 7090 or 7094 computer, coupled with an IBM 1401 computer. The program could be used, with slight modifications, on the Burroughs 220 computer. The Kansas Geological Survey will make the program available, in punched-card form, for a limited time at a cost of $10.00. IBM 7090 and 7094 computers are currently in widespread use in the United States and access to one of these machines is available at a number of both university and commercial computer centers. Persons wishing to use the program on the IBM 7090 or 7094 should send four magnetic tapes to the Computation Center, Stanford University, Stanford, California, so that the BALGOL compiler system can be recorded on the tape. When this has been done, the program described in this report, as well as many other BALGOL programs, may be readily used on virtually any IBM 7090 or 7094 computer. Peter Carah wrote that part of the program for plotting of *z* values and residual values in the "slice maps," and the matrix inversion procedure was adapted from the Stanford University Computation Center program library.

The program is extremely fast and economical when run on the IBM 7090 or 7094 computer. The program compiles from the BALGOL source deck in 30 seconds. Time required for execution of the program varies, depending upon the number of data points and dimensions of the hypersurface blocks and plotted maps. As an example, 60 seconds 7090 time were required for execution using oil-gravity data described in this report, in which 244 data points were handled, three hypersurface blocks about 5 x 8½ x 9 inches in dimensions were contoured, and 24 "slice maps" were plotted. In addition, about 6 minutes 1401 time were required for printing of the output.

Major Steps in Program

The theory and operation of the program are explained in detail on subsequent pages. However, as an introduction, the principal steps in the program are outlined below:

(1) Read into computer numerical data that control certain operations of the program, such as dimensions of block contoured by computer.

(2) Read in data values, four values (*w*, *x*, *y*, and *z*) for each data point.

(3) Obtain sums for matrix and for column vector used in solution of normal equations.

(4) Calculate constants of equations of hypersurfaces by matrix inversion.

(5) Employing equation constants thus obtained, calculate trend value at each data point and subtract this value from actual *z* value at that point to obtain residual value. Each hypersurface of given degree will have its own trend and residual values at specified data points.

(6) Calculate statistical properties of hypersurfaces, including error measure, percent of total sum of squares, and apportionment of the sums of squares according to linear, quadratic and abbreviated cubic components.

(7) Calculate hypervolumes within hypersurfaces by evaluation of triple integral between limits of block. Divide hyper-

volume by ordinary volume to obtain average value of *z* within block.

(8) Contour intersections of hypersurface with planes that intersect block. The number of planes and their intersect values are specified on the control cards. Any number of horizontal and vertical planes may be contoured. It is generally convenient to contour the top, bottom, and four sides of the block whose limits have been specified in making previous calculations. Contouring is accomplished by substituting progressively changing values of *w*, *x*, or *y* corresponding to the location in space of each point for which a printed character is printed. Values of *w*, *x*, or *y* are substituted in equation describing surface, *z* value for each point is determined, and character to be printed (number, blank space, letter, or other symbol; Table 7) is selected, depending on value that *z* assumes at each point (Appendix *C, Part 2*).

(9) If desired, the original data values and residual values from the three trend hypersurfaces are sorted according to depth and location and are then automatically plotted in a series of horizontal "slice maps" on which the values are plotted within specified depth intervals (Fig. 11; Appendix *C, Part 3*).

Input to Program

After the computer program has been fed into the computer, data cards follow. Three kinds of data cards are used in conjunction with this program: (1) alphabetical and numerical information used for identification purposes, (2) numerical information used to control the operation of the program, and (3) numerical values pertaining to data points. These data are described on subsequent pages. Detailed rules for preparation of data cards, as well as other information concerning BALGOL, are contained in the manual entitled *Burroughs Algebraic Compiler: A Representation of ALGOL for Use with the Burroughs 220 Data-Processing System*, which may be obtained from the Burroughs Corporation, Detroit, Michigan. The rules for data cards are very simple, however, and the most important are: (1) a 5 must be punched in column 1 of each data card, (2) columns 2 to 80 of each card are available for data, and (3) there is no specified format for the data values except that at least one blank column must separate numbers.

Alphanumeric heading.—The first data consists of 72 characters of alphabetic and numerical (alphanumeric) information. Typically, this information might include the name of the area being studied, the name and age of the geologic formation or formations involved, and the name of the person preparing the data. When the program is executed, this information is reproduced at the top of each of the map pages (Appendix *C*) and elsewhere in the output pages, thus providing positive identification. Dollar signs should be placed in columns 2 and 75 of the data card, and the 72 alphanumeric characters (including blank spaces) are placed between the two dollar signs, in columns, 3 to 74.

Control cards.—(1) First control card:

(a) A 5 in column 1 of each card.

(b) An integer specifying whether the data point values are type integer (2222) or type decimal (4444).

(c) An integer specifying the number of data points.

(d) An integer specifying the length, in tenths of an inch, of the *y* dimension of the block that is to be contoured by the computer's printing machine. (Fig. 13).

(e) An integer specifying the *x* dimension of the contoured block in tenths of an inch.

(f) A decimal-point number specifying the *x* intercept of the right side of the block. The value must be expressed in *x*-coordinate units.

(g) A decimal specifying the *x* intercept of left side of block.

(h) A decimal specifying the *y* intercept of front of block. The value must be expressed in *y*-coordinate units.

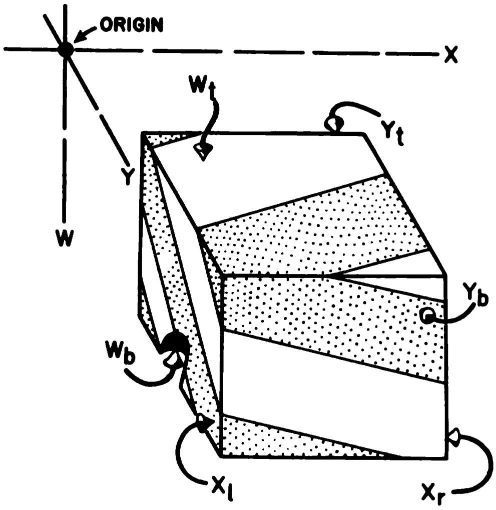

FIGURE 13.—Diagram showing coordinate-axis intercept values (Xr, Xl, Yb, Yt, Wb, and Wt) for six planes that form surfaces of block intersecting hypersurface. (Intercept values must be in same units as w, x, and y values of data points.)

(i) A decimal specifying the y intercept in the back of block.

(j) A decimal specifying the w intercept of bottom of block. The value must be expressed in w-coordinate units.

(k) A decimal specifying the w intercept of top of block.

(l) A decimal specifying the reference contour value.

(m) A decimal specifying the contour interval.

(n) An integer specifying the vertical, or w, dimension, in tenths of an inch, of the block that is to be contoured.

(o) An integer specifying whether trend and residual values are to be written, written and punched, or neither written or punched as follows:

0 Do not print or punch
1 Print only
2 Print and punch

(p) An integer specifying whether raw data and residual values are to be plotted as follows:

0 Do not plot
1 Plot

(2) Second control card (for horizontal contoured surfaces):

(a) A 5 in column 1.

(b) An integer specifying the number of horizontal surfaces to be contoured; punch 0 if none are to be contoured.

(c) Decimal numbers specifying intercept values of horizontal surfaces, expressed in w-axis coordinate units.

(3) Third control card (for vertical surfaces intersecting x axis):

(a) A 5 in column 1.

(b) An integer specifying the number of surfaces to be contoured; 0 if none are to be contoured.

(c) Decimal numbers specifying x-axis intercept values of the surfaces.

(4) Fourth control card (for vertical surfaces intersecting y axis):

(a) A 5 in column 1.

(b) An integer specifying number of surfaces to be contoured; 0 if none are to be contoured.

(c) Decimal numbers specifying y-axis intercept values of surfaces.

(5) Fifth control card (controls residual plotting; use only if integer specifying whether data values are to be plotted is 1 on first control card):

(a) A 5 in column 1.

(b) A decimal specifying the w intercept value of the top "slice" in which the original z values and residual values will be plotted. This number will have a smaller numerical value than in (d) below because the numbers increase going downward.

(c) A decimal specifying the thickness in vertical (w-axis) units of each of the "slices."

(d) A decimal specifying the w intercept value of the top of the bottom "slice." This number will necessarily be greater than the number in (b) above.

Values for data points.—Four values for each data point are needed. These are fed in, in groups of four, in the order *w*, *x*, *y* and *z*. The values may be either wholly decimal-point numbers or wholly integers, but may not be a mixture of both decimal and integer numbers. The *w*, *x*, and *y* values are coordinate values in arbitrary units, which may have a dimensional sense (feet, miles, fractions of inch, etc.), but could also represent any quantity that one wishes. For most geological purposes, it is convenient to express the *w* values in feet (well depth for example) and the *x* and *y* values in either miles or in fractions of an inch scaled from a map. The *z* values may be in any convenient units. If the integer on the first control card that specifies the type of data values is 2222, all values are to be in integer form, if 4444, all values in decimal-point numbers. Please note that a maximum of 950 data points may be handled without modification of the program. However, more data points could be handled by changing the array dimensions on lines 8, 12 and 19, Appendix *B*.

In obtaining the *w*-, *x*-, and *y*-coordinate values, it is most convenient to place the origin either at the upper left rear corner of the block or at some point that is farther to the left, higher, and farther to the rear. In a conventional geological application, the *w*-coordinate values might be well depths, with positive values that increase downward, and the *x*- and *y*-coordinate values might represent distances scaled from an origin along east-west and north-south directions, respectively. Negative values are acceptable. Cards are to be punched as follows:

(a) A 5 in column 1.

(b) *w*, *x*, *y*, and *z* values, in that order (any number of values per card, but numbers will be read in groups of four).

Solutions of Normal Equations to Obtain Constants of Equations

Each hypersurface is described by an equation whose constants are such that the least-squares criterion is satisfied. The method employed involves matrix inversion and is basically the same for each hypersurface, regardless of the number of terms in its equation. For illustrative purposes, only the first-degree hypersurface is considered in detail below.

First-degree hypersurface. — The equation for a first degree hypersurface is

(Eq. A)

$$z_{trend} = A + Bw + Cx + Dy.$$

The constants *A*, *B*, *C* and *D* of this equation are to be calculated so that the sum of the squared deviations is the least possible. The deviation at a particular point is that difference between the observed and calculated value. which may be expressed as

$$\text{deviation} = z_{obs} - z_{trend}.$$

Because the z_{trend} value is given by equation A, we may rewrite equation as follows:

(Eq. B)

$$\text{deviation} = z_{obs} - (A + Bw + Cx + Dy). \text{ or}$$
$$\text{deviation} = z_{obs} - A - Bw - Cx - Dy.$$

Proceeding further, we may express the sum of squared deviations as a function, *F*, of *A*, *B*, *C*, and *D*, by writing

(Eq. C)

$$\text{sum of squared deviations} = F\ (A, B, C, D,).$$

Combining equations *B* and *C*, we obtain

$$F\ (A, B, C, D) = \Sigma\ (z_{obs} - A - Bw - Cx - Dy)^2.$$

From this point on we will consider *z* to be z_{obs}. If *F* (*A*, *B*, *C*, *D*) is to be minimized, it is necessary that

$$\partial F/\partial A = \partial F/\partial B = \partial F/\partial C = \partial F/\partial D = O.$$

The partial derivatives are

$$\frac{\partial F}{\partial A} = \Sigma\ 2(z - A - Bw - Cx - Dy)(-1) = 0$$

$$\frac{\partial F}{\partial B} = \Sigma\ 2(z - A - Bw - Cx - Dy)(-w) = 0$$

$$\frac{\partial F}{\partial C} = \Sigma\ 2(z - A - Bw - Cx - Dy)(-x) = 0$$

$$\frac{\partial F}{\partial D} = \Sigma\ 2(z - A - Bw - Cx - Dy)(-y) = 0.$$

Multiplication of each expression and summation over the individual terms of these four equations yields four other equations, which are

$$-\Sigma z + An + B\Sigma w + C\Sigma x + D\Sigma y = 0$$
$$-\Sigma zw + A\Sigma w + B\Sigma w^2 + C\Sigma wx + D\Sigma wy = 0$$
$$-\Sigma zx + A\Sigma x + B\Sigma wx + C\Sigma x^2 + D\Sigma xy = 0$$
$$-\Sigma zy + A\Sigma y + B\Sigma wy + C\Sigma xy + D\Sigma y^2 = 0$$

where n = number of data points.

We may rearrange these equations by placing the terms containing z on the other side to obtain four normal equations whose solution will permit us to obtain the four unknown constants, $A, B, C,$ and D:

$$An + B\Sigma w + C\Sigma x + D\Sigma y = \Sigma z$$
$$A\Sigma w + B\Sigma w^2 + C\Sigma wx + D\Sigma wy = \Sigma zw$$
$$A\Sigma x + B\Sigma wx + C\Sigma x^2 + D\Sigma xy = \Sigma zx$$
$$A\Sigma y + B\Sigma wy + C\Sigma xy + D\Sigma y^2 = \Sigma zy \; .$$

If a solution exists for these four linear equations, they may usually be solved by matrix algebra methods. We may restate the four normal equations by writing a single matrix equation, as follows:

$$\begin{bmatrix} n & \Sigma w & \Sigma x & \Sigma y \\ \Sigma w & \Sigma w^2 & \Sigma wx & \Sigma wy \\ \Sigma x & \Sigma wx & \Sigma x^2 & \Sigma xy \\ \Sigma y & \Sigma wy & \Sigma xy & \Sigma y^2 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} \Sigma z \\ \Sigma zw \\ \Sigma zx \\ \Sigma zy \end{bmatrix}$$

In this equation, the $ABCD$-vector multiplied by the wxy-matrix is equal to the vector containing z. In applying this equation, observational data provide the wxy-matrix and the z-vector, allowing the $ABCD$-vector to be determined. This may be done by multiplying the z-vector with the inverse of the wxy-matrix, so that

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} n & \Sigma w & \Sigma x & \Sigma y \\ \Sigma w & \Sigma w^2 & \Sigma wx & \Sigma wy \\ \Sigma x & \Sigma wx & \Sigma x^2 & \Sigma xy \\ \Sigma y & \Sigma wy & \Sigma xy & \Sigma y^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \Sigma z \\ \Sigma zw \\ \Sigma zx \\ \Sigma zy \end{bmatrix}$$

Second- and third-degree hypersurfaces.—Second- and third-degree hypersurfaces are fitted in essentially the same manner as first-degree hypersurfaces, and the underlying theory is the same. There are more terms in the equations describing second- and third-degree surfaces, and therefore more normal equations are needed. The matrix equation for combined linear, quadratic, and part of the cubic terms is shown in Table 6. First-degree hypersurfaces involve only the linear terms (outlined by dotted lines in Table 6), second-degree hypersurfaces involve linear plus quadratic terms (outlined with dashed lines), and third-degree hypersurfaces involve all the linear plus quadratic plus cubic terms. It should be pointed out that the cubic terms are not complete in that the cubic cross-product terms have been arbitrarily omitted to cut down on the number of steps in the program.

Steps in computer program in solving normal equations.—The summation to obtain the values for z-vector and the wxy-matrix (Table 6) is accomplished in a FOR loop (lines 55 to 129, Appendix *B*). Inasmuch as many elements in the matrix are duplicates of others, it is not necessary to calculate all of them by summation. Those that are duplicates are simply assigned (lines 130 to 194, Appendix *B*). Because the matrices and column vectors are altered each time they are used in solving the matrix equation, new matrices and new column vectors are assigned using FOR loops (lines 195 to 201) to preserve the original matrices and vectors.

In the program, solution of matrix equations is accomplished by procedure SOLV (lines 204 to 252) and binary external procedure INPROD (lines 753 to 755) which has been declared (line 203) ahead of procedure SOLV. Each time procedure SOLV is called, the identifiers and the dimensions of the matrix and the two vectors are specified (lines 253, 255, and 257. Thus, the same matrix-equation solving technique is used regardless of whether the equation pertains to first-, second-, or third-degree hypersurfaces.

The numerical values of the elements of the wxy-matrix and the z-vector (Table 6), obtained by summation, are part of the output of the program (Appendix *C, Part 1*). Ordinarily these values are not of direct interest, but it may be helpful to scan them to insure that the memory capacity of the computer is

$$\begin{bmatrix}
N & \Sigma w & \Sigma x & \Sigma y & \Sigma x^2 & \Sigma wy & \Sigma xy & \Sigma wx & \Sigma w^2 & \Sigma y^2 & \Sigma x^3 & \Sigma w^3 & \Sigma y^3 \\
\Sigma w & \Sigma w^2 & \Sigma wx & \Sigma wy & \Sigma wx^2 & \Sigma w^2y & \Sigma wxy & \Sigma w^2x & \Sigma w^3 & \Sigma wy^2 & \Sigma wx^3 & \Sigma w^4 & \Sigma wy^3 \\
\Sigma x & \Sigma wx & \Sigma x^2 & \Sigma xy & \Sigma x^3 & \Sigma wxy & \Sigma x^2y & \Sigma wx^2 & \Sigma w^2x & \Sigma xy^2 & \Sigma x^4 & \Sigma xw^3 & \Sigma xy^3 \\
\Sigma y & \Sigma wy & \Sigma xy & \Sigma y^2 & \Sigma x^2y & \Sigma wy^2 & \Sigma xy^2 & \Sigma wxy & \Sigma w^2y & \Sigma y^3 & \Sigma x^3y & \Sigma w^3y & \Sigma y^4 \\
\Sigma x^2 & \Sigma wx^2 & \Sigma x^3 & \Sigma x^2y & \Sigma x^4 & \Sigma wx^2y & \Sigma x^3y & \Sigma wx^3 & \Sigma w^2x^2 & \Sigma x^2y^2 & \Sigma x^5 & \Sigma w^3x^2 & \Sigma x^2y^3 \\
\Sigma wy & \Sigma w^2y & \Sigma wxy & \Sigma wy^2 & \Sigma wx^2y & \Sigma w^2y^2 & \Sigma wxy^2 & \Sigma w^2xy & \Sigma w^3y & \Sigma wy^3 & \Sigma wyx^3 & \Sigma w^4y & \Sigma wy^4 \\
\Sigma xy & \Sigma wxy & \Sigma x^2y & \Sigma xy^2 & \Sigma x^3y & \Sigma wxy^2 & \Sigma x^2y^2 & \Sigma wx^2y & \Sigma w^2xy & \Sigma xy^3 & \Sigma x^4y & \Sigma w^3xy & \Sigma xy^4 \\
\Sigma wx & \Sigma w^2x & \Sigma wx^2 & \Sigma wxy & \Sigma wx^3 & \Sigma w^2xy & \Sigma wx^2y & \Sigma w^2x^2 & \Sigma w^3x & \Sigma wxy^2 & \Sigma wx^4 & \Sigma w^4x & \Sigma wxy^3 \\
\Sigma w^2 & \Sigma w^3 & \Sigma w^2x & \Sigma w^2y & \Sigma w^2x^2 & \Sigma w^3y & \Sigma w^2xy & \Sigma w^3x & \Sigma w^4 & \Sigma w^2y^2 & \Sigma w^2x^3 & \Sigma w^5 & \Sigma w^2y^3 \\
\Sigma y^2 & \Sigma wy^2 & \Sigma xy^2 & \Sigma y^3 & \Sigma x^2y^2 & \Sigma wy^3 & \Sigma xy^3 & \Sigma wxy^2 & \Sigma w^2y^2 & \Sigma y^4 & \Sigma x^3y^2 & \Sigma w^3y^2 & \Sigma y^5 \\
\Sigma x^3 & \Sigma wx^3 & \Sigma x^4 & \Sigma x^3y & \Sigma x^5 & \Sigma wx^3y & \Sigma x^4y & \Sigma wx^4 & \Sigma w^2x^3 & \Sigma x^3y^2 & \Sigma x^6 & \Sigma w^3x^3 & \Sigma x^3y^3 \\
\Sigma w^3 & \Sigma w^4 & \Sigma w^3x & \Sigma w^3y & \Sigma w^3x^2 & \Sigma w^4y & \Sigma w^3xy & \Sigma w^4x & \Sigma w^5 & \Sigma w^3y^2 & \Sigma w^3x^3 & \Sigma w^6 & \Sigma w^3y^3 \\
\Sigma y^3 & \Sigma wy^3 & \Sigma xy^3 & \Sigma y^4 & \Sigma x^2y^3 & \Sigma wy^4 & \Sigma xy^4 & \Sigma wxy^3 & \Sigma w^2y^3 & \Sigma y^5 & \Sigma x^3y^3 & \Sigma w^3y^3 & \Sigma y^6
\end{bmatrix}
\cdot
\begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \\ K \\ L \\ M \end{bmatrix}
=
\begin{bmatrix} \Sigma z \\ \Sigma wz \\ \Sigma xz \\ \Sigma yz \\ \Sigma x^2z \\ \Sigma wyz \\ \Sigma xyz \\ \Sigma wxz \\ \Sigma w^2z \\ \Sigma y^2z \\ \Sigma x^3z \\ \Sigma w^3z \\ \Sigma y^3z \end{bmatrix}$$

TABLE 6.—Matrix equation for obtaining constants of hypersurface equations. Left column vector contains constants *A* to *M*; Square matrix contains *w*, *x*, and *y* elements; right column vector contains *w*, *x*, *y*, and *z* elements. First-degree matrix equation is contained within dotted lines; second-degree matrix equation is contained within dashed lines (includes first-degree elements); abbreviated third-degree matrix equation contains all elements.

not being exceeded, particularly if very large numbers for data values are involved. The equation constants are part of the output of the program (Appendix *C, Part 1*).

Calculation of Trend and Residual Values

Trend and residual values are calculated for each data point employing a FOR loop (lines 259 to 275, Appendix *B*). The trend values are calculated successively for first-, second-, and third-degree hypersurfaces, using the appropriate equation constants calculated previously. The residual (or deviation) value at each data point is obtained by simply subtracting the trend value from the observed value. If the trend value is algebraically smaller than the observed value, the residual is positive, whereas if the trend value is algebraically greater, the residual is negative.

The *w*-, *x*-, and *y*-coordinate values, observed *z* values, and first-, second-, and third-degree trend and residual values are printed out in a table of ten columns (Appendix *C, Part 1*).

Calculation of Statistical Measures

Error measure.—Error measure (lines 276 to 281 is defined as the sum of the squared residual values, divided by the number of data points, less one, which may be expressed as follows:

$$EM = \frac{\Sigma(z_{obs} - z_{trend})^2}{n - 1}$$

Error measure is thus a measure of the degree to which the calculated trend approaches the observed data values. A perfectly fitted trend would have an error measure of zero.

Sum of Squares.—The sums of squares associated with the linear, quadratic, and abbreviated cubic trend components, and with deviations from these components are calculated (lines 286 to 297, 305). These values may be used to determine confidence levels associated with the components by analysis of variance.

Percent of total sum of squares.—Another measure of the degree to which the trend approaches the observed data is the percent of total sum of squares (lines 298 to 304). The percent of total sum of squares may be defined algebraically as:

$$100 \left\{ \frac{\Sigma z^2_{trend} - \left[\frac{(\Sigma z_{trend})^2}{n}\right]}{\Sigma z^2_{obs} - \left[\frac{(\Sigma z_{obs})^2}{n}\right]} \right\}$$

The percent of total sum of squares may vary from only a few percent to almost 100 percent. A value of 100 percent would indicate a perfect fit of the trend to the observed data.

Calculation of Hypervolumes and Average *z* Value

The program provides for calculation of four-dimensional hypervolumes within the hypersurfaces between specified limits. If four-dimensional hypervolume is divided by three-dimensional volume, an average value of *z* is obtained in which the spatial locations of the *z* data values weight or influence the average. A spatially weighted average calculated in this manner, may in some cases, be more meaningful than the conventional arithmetic mean, particularly where the data values are very irregular and contain extremes. A suggested geological application would be in calculating average porosity values in limestone oil reservoirs, where porosity values obtained by core analysis may be highly erratic.

To explain the principle of this method. analogies have been drawn between calculation of weighted averages where two, three, and four variables are involved (Fig. 14). Consider the problem of obtaining a weighted average of *z*, where *z* is a function of *x*. We may represent the function by a curve (Fig. 14*A*), and if we wish to calculate the average value of *z* between limits x_1 and x_2, we find the area beneath the curve between these limits and then divide by the distance between x_1 and x_2. Inasmuch as *z* is expressed as a function of *x*, the area beneath the curve is obtained by evaluating the integral of the function between the limits x_1 and x_2. Thus, the average of z is the average height of the curve above the x axis between the limits.

Consider now the problem of calculating a weighted average of z where three variables are involved, and z may be expressed as a function of x and y. Whitten (1962) has discussed the theory of the method in detail. Inasmuch as three variables are involved, a surface (Fig. 14*B*) rather than a line represents z as a function of x and y. We may think of the weighted average value of z as being the average height of the surface above the x-y plane between the specified limits, x_1 to x_2, and y_1 to y_2. To obtain the weighted average we calculate the volume between the surface, the x-y plane, and the four planes specified by x_1, x_2, y_1, and y_2. We then divide this volume by the area in the x-y plane within the limits. The volume is obtained by double integration and evaluation of the integral between the limits.

We may now consider the problem of obtaining a weighted average where four variables are involved, and z may be expressed as a function of w, x, and y and is represented by a hypersurface. The volume within a four-dimensional hypersurface is a four-dimensional hypervolume. When the hypervolume is divided by three-dimensional volume, a weighted average of z is obtained. The principles are the same as with a lesser number of variables. The hypervolume is obtained by evaluation of the triple integral (Fig. 14*C*) between the three pairs of limits, w_1 to w_2, x_1 to x_2, and y_1 to y_2.

Hypervolume within a first-degree hypersurface.—The mathematical steps in obtaining hypervolume within a first-degree hypersurface are outlined below. Hypervolumes within higher-degree hypersurfaces are obtained in the same way, except that the equations have more terms.

The hypervolume within a hypersurface is given by the indefinite triple integral

$$\int dw \int dx \int z \; dy \; ,$$

where z is a function of w, x, and y:

$$z = f(w, x, y) \; .$$

For a first-degree hypersurface, the function is

$$z = A + Bw + Cx + Dy \; .$$

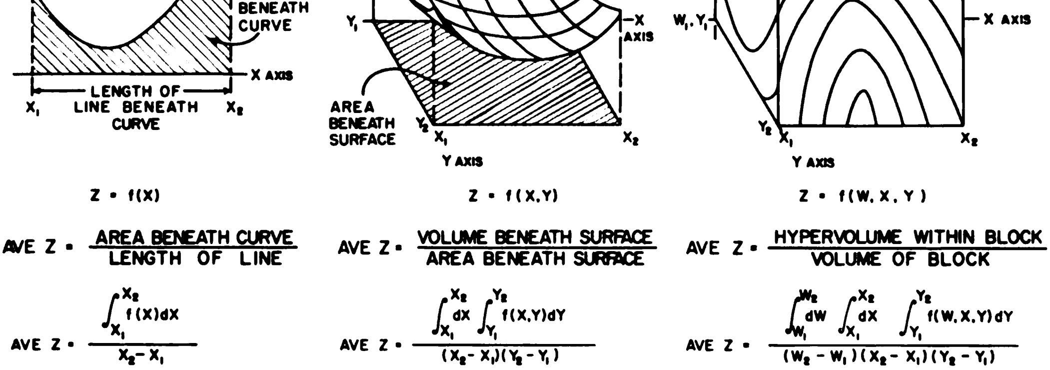

FIGURE 14.—Diagrams and generalized equations showing how spatially-weighted average values may be obtained by integration and division where two (*A*), three (*B*), and four (*C*) variables are involved.

Substituting in this function, we obtain the indefinite triple integral

$$\int dw \int dx \int (A + Bw + Cx + Dy)\, dy \; .$$

Integrating with respect to *y*, we obtain

$$\int dw \; \{ \int [Ay + Bwy + Cxy + \tfrac{1}{2}Dy^2]\, dx \} \; dw.$$

In turn, integrating with respect to *x*, we obtain

$$\int \{ Axy + Bwxy + \tfrac{1}{2}Cx^2y + \tfrac{1}{2}Dxy^2 \} \; dw \; ,$$

and finally, integrating with respect to *w*, we obtain

$$Awxy + \tfrac{1}{2}Bw^2xy + \tfrac{1}{2}Cwx^2y + \tfrac{1}{2}Dwxy^2 \; .$$

We may determine the actual hypervolume by evaluation of the definite integral between specified limits. The general form of the definite triple integral, where *z* is a function of *w*, *x*, and *y*, may be written

$$\int_{w_1}^{w_2} dw \int_{x_1}^{x_2} dx \int_{Y_1}^{Y_2} dy \quad ,$$

in which the limits represent the values of *w*, *x*, and *y* at the edges of the block in which the hypervolume is to be calculated (Fig. 14C). Thus, the hypervolume is obtained when the equation above is evaluated between these limits by substituting the following values for *w*, *x*, and *y* (lines 355, 358, 361):

$$w = w_2 - w_1$$

$$x = x_2 - x_1$$

$$y = y_2 - y_1$$

$$w^2 = w_2^2 - w_1^2$$

$$x^2 = x_2^2 - x_2^2$$

$$y^2 = y_2^2 - y_2^2 \; .$$

The hypervolume between these limits (lines 361 to 365) is given by

$$\begin{aligned} \text{Volume} = \; & A[(w_2 - w_1)(x_2 - x_1)(y_2 - y_1)] \\ & + \tfrac{1}{2}B[(w_2^2 - w_1^2)(x_2 - x_1)(y_2 - y_1)] \\ & + \tfrac{1}{2}C[(w_2 - w_1)(x_2^2 - x_1^2)(y_2 - y_1)] \\ & + \tfrac{1}{2}D[(w_2 - w_1)(x_2 - x_1)(y_2^2 - y_1^2)] \end{aligned}$$

Hypervolume within second- and third-degree hypersurfaces.—The volume within a second-degree hypersurface (lines 355, 356, 358, 359, 361, 362, 367 to 372) is given by the equation:

$$\begin{aligned} \text{Volume} = \; & A[(w_2 - w_1)(x_2 - x_1)(y_2 - y_1)] \\ & + 1/2\; B[(w_2^2 - w_1^2)(x_2 - x_1)(y_2 - y_1)] \\ & + 1/2\; C[(w_2 - w_1)(x_2^2 - x_1^2)(y_2 - y_1)] \\ & + 1/2\; D[(w_2 - w_1)(x_2 - x_1)(y_2^2 - y_1^2)] \\ & + 1/3\; E[(w_2 - w_1)(x_2^3 - x_1^3)(y_2 - y_1)] \\ & + 1/3\; F[(w_2 - w_1)(x_2 - x_1)(y_2^3 - y_1^3)] \\ & + 1/3\; G[(w_2^3 - w_1^3)(x_2 - x_1)(y_2 - y_1)] \\ & + 1/4\; H[(w_2 - w_1)(x_2^2 - x_1^2)(y_2^2 - y_1^2)] \\ & + 1/4\; I[(w_2^2 - w_1^2)(x_2^2 - x_1^2)(y_2 - y_1)] \\ & + 1/4\; J[(w_2^2 - w_1^2)(x_2 - x_1)(y_2^2 - y_1^2)] \end{aligned}$$

The expression for hypervolume within a third-degree hypersurface (lines 354 to 363, 374 to 379) is not given here. Cubic cross-product terms have been omitted in the program for the third-degree hypersurface, cutting down substantially on the number of arithmetic operations in evaluating the expression.

Contouring of Intersection of Hypersurface with a Plane

A hypersurface can be visualized by passing planes through it and contouring the values of the hypersurface where they intersect the planes. In this program, planes are contoured one at a time (Appendix *C, Part 2*) and may be pasted together later to form a rectangular block (Fig. 15).

Contouring (lines 426 to 689) is accomplished by holding constant the coordinate value (w, x, or y) that specifies the plane, progressively varying the other two coordinates values, and solving for z with the equation describing the hypersurface. Values assigned to the two coordinates are progressively changed according to the spacing in columns and rows of the printed characters of the computer's printing machine (an IBM 1403 high-speed printer has been used in the examples shown).

After the value of z has been determined at a particular point on the plane on which the contours are being drawn, either a blank space or a certain character, which may be a number, letter, or other character (Table 7) is printed. The character printed depends on the value of z at that point, the reference contour value, and the contour interval value. The printed characters have been arranged so that there is little likelihood of ambiguity. The steps in calculating the values of contour lines are as follows:

(A) Determine the number of contour intervals represented by the band of characters or blanks by referring to Table 7.

(B) Multiply this number by the contour interval.

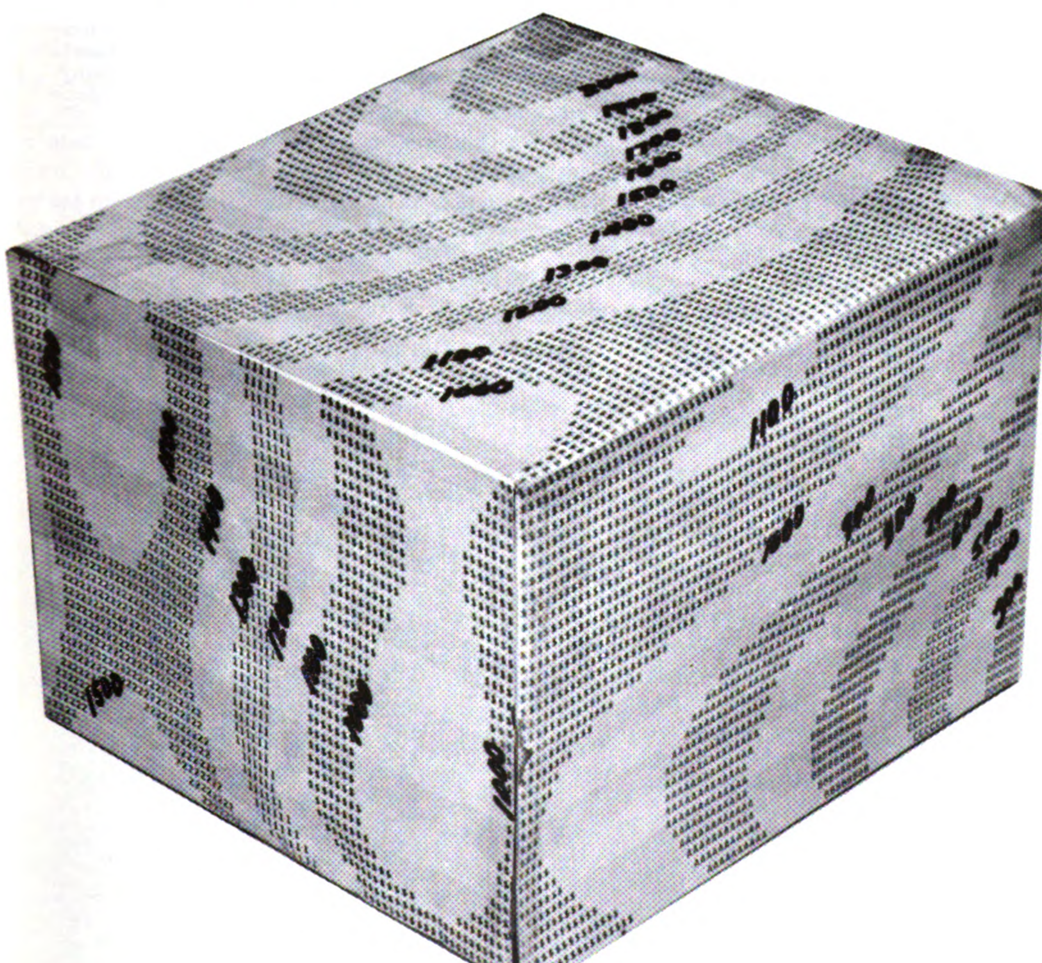

FIGURE 15.—Hypersurface block constructed by pasting together surfaces on which contour bands have been printed by computer's printing machine. Numbers denoting values of edges of bands were put on by hand.

(C) Add algebraically to the reference contour value.

For example, if the contour value is 100, and the contour interval is 10, then the contour value represented by the algebraically lower edge of the band printed with B's is 60. The reference contour value is marked by the edge of the band of $'s which faces the band of A's.

Plotting of Original Data and of Residual Values

In this program, all plotting of data points is done automatically. It is possible to do this because the location of each data point is specified by the coordinate values *w*, *x*, and *y*. Plotting of points on an ordinary map is usually no problem because the location of any point can be specified by two coordinate values, *x* and *y*. But, plotting of points in three-dimensional space poses a problem. In this program (lines 690 to 750) points are plotted by dividing the three-dimensional block (Fig. 11*A*) into a series of horizontal slices (Appendix *C*, *Part 3*), each slice being of specified constant thickness. Data points

TABLE 7.—List of characters that correspond with contour intervals of printed contour maps. Empty places in column indicate that no character is printed.

Number of contour intervals above (+) or below (−) reference contour.	Character printed (or blank) in band, of which lower edge denotes contour value.	Number of contour intervals above (+) or below (−) reference contour.	Character printed (or blank) in band, of which lower edge denotes contour value.
−40	T	+ 1	
−39		+ 2	1
−38	S	+ 3	
−37		+ 4	2
−36	R	+ 5	
−35		+ 6	3
−34	Q	+ 7	
−33		+ 8	4
−32	P	+ 9	
−31		+10	5
−30	O	+11	
−29		+12	6
−28	N	+13	
−27		+14	7
−26	M	+15	
−25		+16	8
−24	L	+17	
−23		+18	9
−22	K	+19	
−21		+20	0
−20	J	+21	
−19		+22	$
−18	I	+23	
−17		+24	*
−16	H	+25	
−15		+26	−
−14	G	+27	
−13		+28	.
−12	F	+29	
−11		+30	+
−10	E	+31	
− 9		+32	=
− 8	D	+33	
− 7		+34	W
− 6	C	+35	
− 5		+36	X
− 4	B	+37	
− 3		+38	Y
− 2	A	+39	
− 1			
0	$		

within a slice are projected onto a plane and plotted as points on an ordinary map. The number of maps plotted equals the number of slices. This approach does not completely avoid the difficulty of plotting points in three-dimensional space because differences in elevation of points within a slice are ignored. However, it is a practical approach to a problem that has no simple solution.

In the program the values must be sorted before they are plotted. First, the values are sorted according to the *w*-coordinate values, which in a subsurface geological application would be according to depth. The values are then assigned to appropriate slices, sorted according to *y*-coordinate values, and finally sorted according to *x*-coordinate values. The data points within each slice are plotted at the approximate locations specified by their *x*- and *y*-coordinate values. The number printed for each point is located so that its right edge generally corresponds with the actual map location of the data point. Spaces between points are left blank. Of course, small errors are introduced because the numbers are confined to the printer's columns and rows, which are spaced 1/10 and 1/6 of an inch apart, respectively. Additional errors may be introduced where the points are very close together so that the printed numbers tend to overlap. Because all the characters in each line or row are printed simultaneously, printed characters cannot overlap. To avoid this mechanical problem, the location of each printed point is shifted to the right where it would tend to overlap the number representing the data point immediately to the left. At least one blank space is left between numbers, except where a minus sign is present. All numbers are truncated to integers to save space in printing.

As noted previously, thickness of the slices and the *w*-coordinate values of the upper surfaces of the uppermost and the lowermost slice must be specified in the input data. In the program, the original data values are plotted first, followed successively by the first-, second-, and third-degree residual values.

APPENDIX *B*

LISTING OF COMPUTER PROGRAM FOR FITTING HYPERSURFACES

Each line has been arbitrarily numbered for identification at left edge of the line. In actual practice identification numbers are confined to columns 73 to 80 on the punch cards, and 2 is necessary in column 1.

```
COMMENT PROGRAM 15, FITS 4TH DIMENSIONAL 1ST, 2ND, AND 3RD DEGREE
VOLUMES BY LEAST SQUARES TO IRREGULARLY-SPACED DATA POINTS. VOLUMES
CALCULATED BY TRIPLE INTEGRATION AND HORIZONTAL SURFACES ARE CONTOURED.
J.W. HARBAUGH, GEOLOGY DEPT., STANFORD $
INTEGER  XP(), I, J, K, L, OP, N, VERT, HOR,   CV(), EL, M,W $
INTEGER ARMIN, ARPLS , YDIM, WDIM , LY, LX , DIGITS, N2  $
INTEGER PLTAR, PRINT, IY, IX, C= TEMP, LINE   $
ARRAY  PRINT(3,50), IY(950), IX(950) , DIGITS(50) $
ARRAY              PLTAR(12)=''Z VALU','ES ORI','G DATA','1STDEG',
           'REE RE','SIDUAL','2NDDEG','REE RE','SIDUAL','3RDDEG',
           'REE RE','SIDUAL')$
INTEGER I2 $ ARRAY  I2(950)$
ARRAY ARMIN(40) = (' ','A',' ','B',' ','C',' ','D',' ','E',' ','F',
          ' ','G',' ','H',' ','I',' ','J',' ','K',' ','L',' ','M',' ','N'
    ,' ','O',' ','P',' ','Q',' ','R',' ','S',' ','T')  $
ARRAY ARPLS(40) = ('$',' ','1',' ','2',' ','3',' ','4',' ','5',' ',
    '6',' ','7',' ','8',' ','9',' ','0',' ','$',' ','*',' ','-',' ',
    '.',' ','+',' ','=',' ','W',' ','X',' ','Y',' ') $
ARRAY X(950,10),T(13,13), R(13), XP(950,4), T4(6,6), T10(12,12),
    T13(15,15), R4(4), R10(10), R13(13), Q(4), S(10), F(13), Z(132),
    CV(132), LEV(20), LEX(20), LAY(20) $
FORMAT FORM(SKS(BSPRINT(3,I)S,IS DIGITS(I)S),W)$
OUTPUT PLOT(PLTAR((1.5(C-1))-3.5), PLTAR((1.5(C-1))-2.5),
    PLTAR(1.5(C-2))) $
FORMAT PLFT(3A6,W,W) $
FORMAT RESLEV( *LOWER LEVEL OF SLICE = *,X8.2,*     UPPER LEVEL *,
    *OF SLICE =    *,  X8.2, W,W)$
OUTPUT RESOUT ( FWSTEP,  FW)$
START..
INPUT ALPH(A1,A2,A3,A4,A5,A6,A7,A8,A9,A10,A11,A12)       $
READ ($$ ALPH) $
INPUT PREF( OP, N, YDIM, HOR, XR, XL, YB, YT, WB, WT, RF, CON,
WDIM, PUNCHOP, RESIDOP)$
READ ($$ PREF)$
INPUT ELVS(EL,FOR I = (1,1,EL) $ LEV(I))$  READ ($$ ELVS)$
INPUT LXVS(LX,FOR I = (1,1,LX) $ LEX(I))$  READ ($$ LXVS)$
INPUT LYVS(LY,FOR I = (1,1,LY) $ LAY(I))$  READ ($$ LYVS)$
IF RESIDOP EQL 1 $(INPUT RPLOTCON(LOW,STEP,HIGH)$ READ($$RPLOTCON))$
YDAM = YDIM $ HAR  = HOR $
IF OP EQL 2222 $
    BEGIN
        INPUT DATI(FOR I =(1,1,N) $ FOR J =(1,1,4) $ XP(I,J))$
        READ ($$ DATI) $
        FOR I =(1,1,N) $ FOR J =(1,1,4) $ X(I,J) = XP(I,J) $
    END $
IF OP EQL 4444 $
    BEGIN
        INPUT DATD(FOR I =(1,1,N) $ FOR J =(1,1,4) $ X(I,J)) $
        READ ($$ DATD) $
    END $
FOR K =(1,1,13) $ FOR L ='1,1,13) $ T(K,L) = 0.0 $
FOR K =(1,1,13) $ R(K) = 0.0 $
COMMENT CALCULATE VALUES FOR 13 X 13 T MATRIX ELEMENTS, EXCEPT FO-
THOSE VALUES THAT ARE DUPLICATES OF OTHERS,WHICH WILL BE ASSIGNED $
FOR I =(1,1,N) $
BEGIN
T(1,2) = T(1,2) + X(I,1) $
T(1,3) = T(1,3) + X(I,2) $
T(1,4) = T(1,4) + X(I,3) $
T(1,5) = T(1,5) +(X(I,2).X(I,2))$
T(1,6) = T(1,6) +(X(I,1).X(I,3))$
T(1,7) = T(1,7) +(X(I,2).X(I,3))$
T(1,8) = T(1,8) +(X(I,1).X(I,2))$
```

```
T(1,9)  = T(1,9)  +(X(I,1).X(I,1))$
T(1,10)= T(1,10)+(X(I,3).X(I,3))$
T(2,5)  = T(2,5)  +(X(I,1).(X(I,2).X(I,2))) $
T(2,6)  = T(2,6)  +(X(I,1).(X(I,1).X(I,3))) $
T(2,7)  = T(2,7)  +(X(I,1).(X(I,2).X(I,3))) $
T(2,8)  =  T(2,8)  +(X(I,1).(X(I,1).X(I,2))) $
T(2,9)  =  T(2,9)  +(X(I,1).(X(I,1).X(I,1))) $
T(2,10)=  T(2,10)+(X(I,1).(X(I,3).X(I,3))) $
T(3,5)  =  T(3,5)  +(X(I,2).(X(I,2).X(I,2))) $
T(3,7)  =  T(3,7)  +(X(I,2).(X(I,2).X(I,3))) $
T(3,10)=  T(3,10)+(X(I,2).(X(I,3).X(I,3))) $
T(4,10)=  T(4,10)+(X(I,3).(X(I,3).X(I,3))) $
T(5,5)  =  T(5,5)  + ((X(I,2).X(I,2)).(X(I,2).X(I,2))) $
T(5,6)  =  T(5,6)  + ((X(I,1).X(I,2)).(X(I,2).X(I,3))) $
T(5,7)  =  T(5,7)  + ((X(I,2).X(I,2)).(X(I,2).X(I,3))) $
T(5,8)  =  T(5,8)  + ((X(I,1).X(I,2)).(X(I,2).X(I,2))) $
T(5,9)  =  T(5,9)  + ((X(I,1).X(I,1)).(X(I,2).X(I,2))) $
T(5,10)=  T(5,10)+ ((X(I,2).X(I,2)).(X(I,3).X(I,3))) $
T(6,6 )=  T(6,6 )+ ((X(I,1).X(I,1)).(X(I,3).X(I,3))) $
T(6,7 )=  T(6,7 )+ ((X(I,1).X(I,2)).(X(I,3).X(I,3))) $
T(6,8 )=  T(6,8 )+ ((X(I,1).X(I,1)).(X(I,2).X(I,3))) $
T(6,10)=  T(6,10)+ ((X(I,1).X(I,3)).(X(I,3).X(I,3))) $
T(7,10)=  T(7,10)+ ((X(I,2).X(I,3)).(X(I,3).X(I,3))) $
T(8,9 )=  T(8,9 )+ ((X(I,1).X(I,1)).(X(I,1).X(I,2))) $
T(6,9 )=  T(6,9 )+ ((X(I,1).X(I,1)).(X(I,1).X(I,3))) $
T(9,9 )=  T(9,9 )+ ((X(I,1).X(I,1)).(X(I,1).X(I,1))) $
T(9,10)=  T(9,10)+ ((X(I,1).X(I,1)).(X(I,3).X(I,3))) $
T(10,10)=T(10,10)+ ((X(I,3).X(I,3)).(X(I,3).X(I,3))) $
T( 5,11)=T( 5,11)+((X(I,2).X(I,2)).(X(I,2).(X(I,2).X(I,2)))) $
T( 5,12)=T( 5,12)+((X(I,1).X(I,1)).(X(I,1).(X(I,2).X(I,2)))) $
T( 5,13)=T( 5,13)+((X(I,2).X(I,2)).(X(I,3).(X(I,3).X(I,3)))) $
T( 6,11)=T( 6,11)+((X(I,1).X(I,2)).(X(I,2).(X(I,2).X(I,3)))) $
T( 6,12)=T( 6,12)+((X(I,1).X(I,1)).(X(I,1).(X(I,1).X(I,3)))) $
T( 6,13)=T( 6,13)+((X(I,1).X(I,3)).(X(I,3).(X(I,3).X(I,3)))) $
T( 7,11)=T( 7,11)+((X(I,2).X(I,2)).(X(I,2).(X(I,2).X(I,3)))) $
T( 7,12)=T( 7,12)+((X(I,1).X(I,1)).(X(I,1).(X(I,2).X(I,3)))) $
T( 7,13)=T( 7,13)+((X(I,2).X(I,3)).(X(I,3).(X(I,3).X(I,3)))) $
T( 8,11)=T( 8,11)+((X(I,1).X(I,2)).(X(I,2).(X(I,2).X(I,2)))) $
T( 8,12)=T( 8,12)+((X(I,1).X(I,1)).(X(I,1).(X(I,1).X(I,2)))) $
T( 8,13)=T( 8,13)+((X(I,1).X(I,2)).(X(I,3).(X(I,3).X(I,3)))) $
T( 9,11)=T( 9,11)+((X(I,1).X(I,1)).(X(I,2).(X(I,2).X(I,2)))) $
T( 9,12)=T( 9,12)+((X(I,1).X(I,1)).(X(I,1).(X(I,1).X(I,1)))) $
T( 9,13)=T( 9,13)+((X(I,1).X(I,1)).(X(I,3).(X(I,3).X(I,3)))) $
T(10,11)=T(10,11)+((X(I,2).X(I,2)).(X(I,2).(X(I,3).X(I,3)))) $
T(10,12)=T(10,12)+((X(I,1).X(I,1)).(X(I,1).(X(I,3).X(I,3)))) $
T(10,13)=T(10,13)+((X(I,3).X(I,3)).(X(I,3).(X(I,3).X(I,3)))) $
T(11,11)=T(11,11)+((X(I,2).(X(I,2).X(I,2))).(X(I,2).(X(I,2).X(I,2)))) $
T(11,12)=T(11,12)+((X(I,1).(X(I,1).X(I,1))).(X(I,2).(X(I,2).X(I,2)))) $
T(11,13)=T(11,13)+((X(I,2).(X(I,2).X(I,2))).(X(I,3).(X(I,3).X(I,3)))) $
T(12,12)=T(12,12)+((X(I,1).(X(I,1).X(I,1))).(X(I,1).(X(I,1).X(I,1)))) $
T(12,13)=T(12,13)+((X(I,1).(X(I,1).X(I,1))).(X(I,3).(X(I,3).X(I,3)))) $
T(13,13)=T(13,13)+((X(I,3).(X(I,3).X(I,3))).(X(I,3).(X(I,3).X(I,3)))) $
R(1 ) = R(1 ) +  X(I,4)                          $
R(2 ) = R(2 ) + (X(I,4).X(I,1))                  $
R(3 ) = R(3 ) + (X(I,4).X(I,2))                  $
R(4 ) = R(4 ) + (X(I,4).X(I,3))                  $
R(5 ) = R(5 ) + (X(I,4).(X(I,2).X(I,2)))  $
R(6 ) = R(6 ) + (X(I,4).(X(I,1).X(I,3)))  $
R(7 ) = R(7 ) + (X(I,4).(X(I,2).X(I,3)))  $
R(8 ) = R(8 ) + (X(I,4).(X(I,1).X(I,2)))  $
R(9 ) = R(9 ) + (X(I,4).(X(I,1).X(I,1)))  $
R(10) = R(10) + (X(I,4).(X(I,3).X(I,3)))  $
R(11) = R(11) + ((X(I,4).X(I,2)).(X(I,2).X(I,2))) $
```

```
R(12) = R(12) + ((X(I,4).X(I,1)).(X(I,1).X(I,1))) $
R(13) = R(13) + ((X(I,4).X(I,3)).(X(I,3).X(I,3))) $
END $
T( 1,1 ) =  N  $
T( 2,1 ) =  T( 1,2 ) $   T( 1,11) =  T( 3,5 ) $
T( 2,2 ) =  T( 1,9 ) $   T( 1,12) =  T( 2,9 ) $
T( 2,3 ) =  T( 1,8 ) $   T( 1,13) =  T( 4,10) $
T( 2,4 ) =  T( 1,6 ) $   T( 2,11) =  T( 5,8 ) $
T( 3,1 ) =  T( 1,3 ) $   T( 2,12) =  T( 9,9 ) $
T( 3,2 ) =  T( 1,8 ) $   T( 2,13) =  T( 6,10) $
T( 3,3 ) =  T( 1,5 ) $   T( 3,11) =  T( 5,5 ) $
T( 3,4 ) =  T( 1,7 ) $   T( 3,12) =  T( 8,9 ) $
T( 3,6 ) =  T( 2,7 ) $   T( 3,13) =  T(10,7 ) $
T( 3,8 ) =  T( 2,5 ) $   T( 4,11) =  T( 5,7 ) $
T( 3,9 ) =  T( 2,8 ) $   T( 4,12) =  T( 6,9 ) $
T( 4,1 ) =  T( 1,4 ) $   T( 4,13) =  T(10,10) $
T( 4,2 ) =  T( 1,6 ) $   T(11,1 ) =  T( 1,11) $
T( 4,3 ) =  T( 1,7 ) $   T(11,2 ) =  T( 2,11) $
T( 4,4 ) =  T( 1,10) $   T(11,3 ) =  T( 3,11) $
T( 4,5 ) =  T( 3,7 ) $   T(11,4 ) =  T( 4,11) $
T( 4,6 ) =  T( 2,10) $   T(11,5 ) =  T( 5,11) $
T( 4,7 ) =  T( 3,10) $   T(11,6 ) =  T( 6,11) $
T( 4,8 ) =  T( 2,7 ) $   T(11,7 ) =  T( 7,11) $
T( 4,9 ) =  T( 2,6 ) $   T(11,8 ) =  T( 8,11) $
T( 5,1 ) =  T( 1,5 ) $   T(11,9 ) =  T( 9,11) $
T( 5,2 ) =  T( 2,5 ) $   T(11,10) =  T(10,11) $
T( 5,3 ) =  T( 3,5 ) $   T(12,1 ) =  T( 1,12) $
T( 5,4 ) =  T( 3,7 ) $   T(12,2 ) =  T( 2,12) $
T( 6,1 ) =  T( 1,6 ) $   T(12,3 ) =  T( 3,12) $
T( 6,2 ) =  T( 2,6 ) $   T(12,4 ) =  T( 4,12) $
T( 6,3 ) =  T( 2,7 ) $   T(12,5 ) =  T( 5,12) $
T( 6,4 ) =  T( 2,10) $   T(12,6 ) =  T( 6,12) $
T( 6,5 ) =  T( 5,6 ) $   T(12,7 ) =  T( 7,12) $
T( 7,1 ) =  T( 1,7 ) $   T(12,8 ) =  T( 8,12) $
T( 7,2 ) =  T( 2,7 ) $   T(12,9 ) =  T( 9,12) $
T( 7,3 ) =  T( 3,7 ) $   T(12,10) =  T(10,12) $
T( 7,4 ) =  T( 3,10) $   T(13,1 ) =  T( 1,13) $
T( 7,5 ) =  T( 5,7 ) $   T(13,2 ) =  T( 2,13) $
T( 7,6 ) =  T( 6,7 ) $   T(13,3 ) =  T( 7,10) $
T( 7,7 ) =  T( 5,10) $   T(13,4 ) =  T( 4,13) $
T( 7,8 ) =  T( 5,6 ) $   T(13,5 ) =  T( 5,13) $
T( 7,9 ) =  T( 6,8 ) $   T(13,6 ) =  T( 6,13) $
T( 8,1 ) =  T( 1,8 ) $   T(13,7 ) =  T( 7,13) $
T( 8,2 ) =  T( 2,8 ) $   T(13,8 ) =  T( 8,13) $
T( 8,3 ) =  T( 2,5 ) $   T(13,9 ) =  T( 9,13) $
T( 8,4 ) =  T( 2,7 ) $   T(13,10) =  T(10,13) $
T( 8,5 ) =  T( 5,8 ) $   T(12,11) =  T(11,12) $
T( 8,6 ) =  T( 6,8 ) $   T(13,11) =  T(11,13) $
T( 8,7 ) =  T( 5,6 ) $   T(13,12) =  T(12,13) $
T( 8,8 ) =  T( 5,9 )   $   T(3,13) = T(7,10) $
T( 8,10) =  T( 6,7 )   $
T( 9,1 ) =  T( 1,9 )   $
T( 9,2 ) =  T( 2,9 )   $
T( 9,3 ) =  T( 2,8 )   $
T( 9,4 ) =  T( 2,6 )   $
T( 9,5 ) =  T( 5,9 )   $
T( 9,6 ) =  T( 6,9 )   $
T( 9,7 ) =  T( 6,8 )   $
T( 9,8 ) =  T( 8,9 )   $
T(10,1 ) =  T( 1,10)   $
T(10,2 ) =  T( 2,10)   $
T(10,3 ) =  T( 3,10)   $
T(10,4 ) =  T( 4,10)   $
```

```
T(10,5 ) =  T( 5,10)  $
T(10,6 ) =  T( 6,10)  $
T(10,7 ) =  T( 7,10)  $
T(10,8 ) =  T( 6,7 )  $
T(10,9 ) =  T( 9,10)  $
FOR I =(1,1,4) $ FOR J =(1,1,4) ST4(I,J) = T(I,J) $
FOR I =(1,1,13)$ FOR J =(1,1,13)ST13(I,J)= T(I,J) $
FOR I =(1,1,10)$ FOR J =(1,1,10)ST10(I,J)= T(I,J) $
COMMENT ASSIGN 4, 10,AND 13 PORTIONS OF COLUMN VECTOR R TO NEW VECTORS$
FOR I =(1,1,4) $ R4(I) = R(I) $
FOR I =(1,1,13)$ R13(I) =R(I) $
FOR I =(1,1,10)$ R10(I) =R(I) $
COMMENT SOLVE LINEAR MATRIX EQUATIONS OF GENERAL FORM TS = R $
EXTERNAL PROCEDURE INPROD() $
PROCEDURE SOLV(N$A(,),B(),Y()$SINGULAR)$
BEGIN
COMMENT THIS IS THE METHOD OF CROUT,WITH INTERCHANGES,
   TO SOLVE AY=B FOR Y,GIVEN A AND B.         $
   COMMENT   EXTERNAL PROCEDURE INPROD IS CALLED BY SOLV,
   SO INPROD MUST BE AVAILABLE WHEN SOLV IS CALLED.  $
COMMENT SINGULAR IS THE LABEL OF THE STATEMENT TO WHICH
   SOLV() EXITS IF A() IS SINGULAR$
COMMENT REAL A(,),B(),Y()$
INTEGER I,IMAX,J,K,N$
   FOR K=(1,1,N)$
   BEGIN
      TEMP = 0$
      FOR I=(K,1,N)$
      BEGIN
         XX = A(I,K) - INPROD(1,1,K-1,A(I,),A(,K))$
         A(I,K) = XX   $
         IF ABS(XX) GTR TEMP $
         BEGIN
            TEMP = ABS(XXH  $
            IMAX = I
         END
      END$
      IF TEMP EQL 0.0$
      GO SINGULAR$
    COMMENT WE HAVE FOUND THAT A(IMAX,K) IS THE LARGEST PIVOT IN COL K
      NOW WE INTERCHANGE ROWS K AND IMAX$
    IF IMAX NEQ K$
    BEGIN
      FOR J = (1,1,N)$
      BEGIN
         TEMP=A(K,J)$ A(K,J)=A(IMAX,J)$ A(IMAX,J)=TEMP
      END$
      TEMP=B(K)$B(K)=B(IMAX)$B(IMAX)=TEMP
    END$
    COMMENT NOW FOR THE ELIMINATION$
    DENOM = A(K,K)  $
    FOR I=(K+1,1,N)$
    A(I,K) = A(I,K)/DENOM $
    FOR  J = (K+1,1,N)$
      A(K,J) = A(K,J) - INPROD(1,1,K-1,A(K,),A(,J))   $
    B(K) = B(K) - INPROD(1,1,K-1,A(K,),B())
  END$
   FOR I=(1,1,N)$ Y(I) = A(I,I)$
  COMMENT NOW FOR THE BACK SUBSTITUTION$
  FOR K = (N,-1,1)$
    Y(K) = (B(K) - INPROD(K+1,K+1,N-K,A(K,),Y()))/A(K,K)   $
   RETURN
FND SOLV()$
```

```
SOLV( 4$ T4(,), R4(),  Q() $ MAT6) $
MAT6..
SOLV(10$T10(,),R10(),  S() $ MAT10)$
MAT10..
SOLV(13$T13(,),R13(),  F() $ START)$
COMMENT SUBSTITUTE W,X AND Y   VALUES IN EQUATIONS OF FITTED SURFACES $
FOR I =(1,1,N) $
BEGIN
X(I,5) = Q(1) + (Q(2).X(I,1))+ (Q(3).X(I,2)) + (Q(4).X(I,3)) $
X(I,6) = X(I,4) - X(I,5) $
X(I,7) = S(1) + (S(2).X(I,1))+ (S(3).X(I,2)) + (S(4).X(I,3)) +
    (S(5).(X(I,2).X(I,2))) + (S(6).(X(I,1).X(I,3))) +
    (S(7).(X(I,2).X(I,3))) + (S(8).(X(I,1).X(I,2))) +
    (S(9).(X(I,1).X(I,1))) + (S(10).(X(I,3).X(I,3))) $
X(I,8) = X(I,4) - X(I,7) $
X(I,9) = F(1) + (F(2).X(I,1))+ (F(3).X(I,2)) + (F(4).X(I,3)) +
    (F(5).(X(I,2).X(I,2))) + (F(6).(X(I,1).X(I,3))) +
    (F(7).(X(I,2).X(I,3))) + (F(8).(X(I,1).X(I,2))) +
    (F(9).(X(I,1).X(I,1))) + (F(10).(X(I,3).X(I,3))) +
    ((F(11).X(I,2)).(X(I,2).X(I,2))) + ((F(12).X(I,1)).(X(I,1).X(I,1)))+
    ((F(13).X(I,3)).(X(I,3).X(I,3))) $
X(I,10) = X(I,4) - X(I,9) $
END  $
EM1 = EM2 = EM3 = 0.0 $
FOR I =(1,1,N) $
    BEGIN
        EM1 = EM1 +(X(I,6).X(I,6))$
        EM2 = EM2 +(X(I,8).X(I,8))$
        EM3 = EM3 +(X(I,10).X(I,10))$
    END $
    E1   =   EM1/(N-1) $
    E2   =   EM2/(N-1) $
    E3   =   EM3/(N-1) $
SMLN = LNSQ = SMQD = QDSQ = SMCB = CBSQ = SMZ = ZSQ = 0.0 $
FOR I =(1,1,N) $
    BEGIN
        SMLN = SMLN + X(I,5) $
        LNSQ = LNSQ + (X(I,5).X(I,5))$
        SMQD = SMQD + X(I,7) $
        QDSQ = QDSQ + (X(I,7).X(I,7))$
        SMCB = SMCB + X(I,9) $
        CBSQ = CBSQ + (X(I,9).X(I,9))$
        SMZ  = SMZ  + X(I,4) $
        ZSQ  = ZSQ  + (X(I,4).X(I,4))$
    END $
ZOR =  ZSQ - ((SMZ.SMZ)/(N-1)) $
PTS1 = 100.0((LNSQ-((SMLN.SMLN)/(N-1)))/ZOR)$
PTS2 = 100.0((QDSQ-((SMQD.SMQD)/(N-1)))/ZOR)$
PTS3 = 100.0((CBSQ-((SMCB.SMCB)/(N-1)))/ZOR)$
IF PTS1 LSS 0.0 $ PTS1 = 100.0 + PTS1 $
IF PTS2 LSS 0.0 $ PTS2 = 100.0 + PTS2 $
IF PTS3 LSS 0.0 $ PTS3 = 100.0 + PTS3 $
QDON = QDSQ - LNSQ $  CBON = CB2Q -  QDSQ         $
OUTPUT ERMA(E1 ,E2 ,E3 ,PTS1,PTS2,PTS3)$
FORMAT FMTR(*ERROR MEASURE LINEAR TREND SURFACE = *,X31.2,W,W,
             *ERROR MEASURE QUADRATIC TREND SURFACE = *,X28.2,W,W,
             *ERROR MEASURE CUBIC TREND SURFACE = *,X32.2,W,W,
             *PERCENT TOTAL SUM SQUARES LINEAR SURFACE = *,X25.2,W,W,
             *PERCENT TOTAL SUM SQUARES QUADRATIC SURFACE = *,X22.2,W,W,
             *PERCENT TOTAL SUM SQUARES CUBIC SURFACE = *,X26.2,W,W)$
OUTPUT VAR(LNSQ, EM1, QDSQ, QDON, EM2,       CBSQ, EM3, CBON      ) $
FORMAT FVAR(*SUM OF SQUARES DUE LINEAR COMPONENT = *,X30.2,W4,W,
             *SUM OF SQUARED DEVIATIONS FROM LINEAR = *,X28.2,W,W,
```

```
                *SUM OF SQUARES DUE LINEAR + QUADRATIC COMPONENT = *,X18.2,
            W,W,*SUM OF SQUARES DUE TO QUADRATIC ALONE = *,X28.2,W,W,
                *SUM OF SQUARED DEVIATIONS FROM LINEAR + QUADRATIC = *,
                X16.2,W,W,
                *SUM OF SQUARES DUE LINEAR+QUADRATIC+CUBIC = *,X24.2,W,W,
                *SUM OF SQUARED DEVIATIONS FROM LINEAR+QUADRATIC+CUBIC = *,
                X12.2,W,W,
                *SUM OF SQUARES DUE CUBIC ALONE = *,X35.2,W, W)$
AZM = R(1)/N $
OUTPUT ALPHA(A1,A2,A3,A4,A5,A6,A7,A8,A9,A10,A11,A12)$
OUTPUT ODS1(FOR I =(1,1,N) $ FOR J =(1,1,10)$ X(I,J)) $
FORMAT  FMTA(12A6,W3,W) $
OUTPUT C03(FOR J =(1,1,4) $  Q(J)),
       C06(FOR J =(1,1,10)$  S(J)),
       C10(FOR J =(1,1,13)$  F(J))$
FORMAT CF3(W,*EQUATION COEFFICIENTS *,W,*LINEAR, Z = *,X12.5,*+*,
      X12.7,*W +*,X12.7,*X +*,X12.7,*Y*,W,W),
      CF6(*LIN + QUAD, Z =*,X12.5,*+*,X12.7,*W +*,X12.7,*X +*,X12.7,
      *Y +*,X12.8,*X2 +*,X12.8,*WY +*,X12.8,*XY*,W,W,B10,
      *+*,X12.8,*WX +*,X12.8,*W2 +*,X12.8,*Y2*,W,W),
      CF10(*LIN + QUAD + CUB, Z = *,X12.5,*+*,X12.7,*W +*,X12.7,*X +*,
      X12.7,*Y +*,X12.8,*X2 +*,X12.8,*WY*,W,W,B10,*+*,X12.8,*XY +*,
      X12.8,*WX +*,X12.8,*W2 +*,X12.8,*Y2 +*,X12.8,*X3 +*,X12.8,
      *W3 +*,X12.8,*Y3*,W,W) $
FORMAT HED(W,*WCOORD  *,
                  *XCOORD  YCOORD Z-VALUE 1ST-TRD 1ST-RSD 2ND-TRD 2ND-RSD*,
           * 3RD-TRD 3RD-RSD*,W,W) $
FORMAT FMT1(*5*, X7.1, 9X8.1, W) $
FORMAT FMT2(*5*, X7.1, 9X8.1, C) $
FORMAT FMTAP(*5S*, 12A6, *S*, W) $
WRITE ($$ ALPHA, FMTA) $
FORMAT JIM(*13 X 13 (X,Y) MATRIX VALUES *,W,W) $ WRITE ($$ JIM)$
OUTPUT  TRAY(FOR I=(1,1,13)$ FOR J=(1,1,13) $ T(I,J))$
FORMAT  FTRA(W,13F10.3,W) $
WRITE  ($$ TRAY,FTRA) $
FORMAT JOE(W,*1 X 13 COLUMN VECTOR VALUES*,W,W)$ WRITE ($$ JOE)$
OUTPUT  RAR(FOR I =(1,1,13)$  R(I))$  WRITE ($$ RAR, FTRA) $
WRITE ($$ C03, CF3) $ WRITE ($$ C06, CF6)$ WRITE ($$ C10, CF10) $
WRITE ($$ ERMA, FMTR ) $ WRITE '$$ VAR, FVAR) $
WD1 = WB - WBT$  WD2 = (WB.WB) - (WT.WT) $
WD3 = (WB.(WB.WB)) - (WT.(WT.WT)) $
WD4 = ((WB.WB).(WB.WB)) - ((WT.WT).(WT.WT)) $
XD1 = XR - XL $ XD2 =  (XR.XR) - (XL.XL) $
XD3 = (XR.(XR.XR)) - (XL.(XL.XL)) $
XD4 = ((XR.XR).(XR.XR))  - ((XL.XL).(XL.XL)) $
YD1 = YB - YT $ YD2 =  (YB.YB) - (YT.YT) $
YD3 = (YB.(YB.YB)) - (YT.(YT.YT)) $
YD4 = ((YB.YB).(YB.YB))  - ((YT.YT).(YT.YT)) $
AR = WD1.(XD1.YD1) $
VLN = ((Q(1).WD1).(XD1.YD1)) + (((0.5).Q(2)).(WD2.(XD1.YD1)))
   +(((0.5). Q(3)).(WD1.(XD2.YD1))) + (((0.5). Q(4)).(WD1.(XD1.YD2)))$
VQD = ((S(1).WD1).(XD1.YD1)) + (((0.5).S(2)).(WD2.(XD1.YD1)))
   +(((0.5). S(3)).(WD1.(XD2.YD1))) + (((0.5). S(4)).(WD1.(XD1.YD2)))
   +(((0.3333). S(5)).(WD1.(XD3.YD1))) +(((0.25).S(6)).(WD2.(XD1.YD2)))
   +(((0.25).S(7)).(WD1.(XD2.YD2))) + (((0.25).S(8)).(WD2.(XD2.YD1)))
   +(((0.3333).S(9)).(WD3.(XD1.YD1)))
   +(((0.3333).S(10)).(WD1.(XD1.YD3))) $
VCB = ((F(1).WD1).(XD1.YD1)) + (((0.5).F(2)).(WD2.(XD1.YD1))) +
(((0.5).F(3)).(WD1.(XD2.YD1))) + (((0.5).F(4)).(WD1.(XD1.YD2))) +
(((0.3333).F(5)).(WD1.(XD3.YD1))) + (((0.25).F(6)).(WD2.(XD1.YD2))) +
(((0.25).F(7)).(WD1.(XD2.YD2))) + (((0.25).F(8)).(WD2.(XD2.YD1))) +
(((0.3333).F(9)).(WD3.(XD1.YD1)))+ (((0.3333).F(10)).(WD1.(XD1.YD3))) +
(((0.25).F(11)).(WD1.(XD4.YD1))) + (((0.25).F(12)).(WD4.(XD1.YD1))) +
```

```
(((0.25),F(13)),(WD1,(XD1,YD4))) $
AZ1 =  VLN/AR  $  AZ2  =  VQD/AR  $
AZ3 =  VCB/AR $
OUTPUT VOL(VLN, VQD, VCB,AZM, AZ1, AZ2, AZ3, AR) $
FORMAT FMVL(W,*VOLUME WITHIN  LINEAR SURFACE =*,B8,X12.2,W,W,
               *VOLUME WITHIN  LIN+QUAD SURFACE =*,B6,X12.2,W,W,
               *VOLUME WITHIN  LIN+QUAD+CUB SURFACE =*,B2,X12.2,W,W,
               *ARITH. MEAN Z, = SUM OF Z VALUES/ N   = *,X11.2,W,W,
               *AVERAGE Z VALUE, LINEAR SURFACE =*,B6,X12.2,W,W,
               *AVERAGE Z VALUE, LIN+QUAD SURFACE =*,B4,X12.2,W,W,
               *AVERAGE Z VALUE, LIN+QUAD+CUB SURFACE =*,X12.2,W,W,
               *VOL OF BLOCK IN  CUBED UNITS*,B11,X12.2,W,W) $
WRITE ($$ VOL, FMVL) $
WRITE ($$ ALPHA, FMTAP)$
WRITE ($$ HED) $
IF PUNCHOP EQL 1 $ WRITE ($$ ODS1, FMT1)$
IF PUNCHOP EQL 2 $ WRITE ($$ ODS1, FMT2)$
COMMENT CALCULATE DX AND DY, AND SUBSTITUTE PROGRESSIVELY INCREAS+NG
VALUES OF X AND Y MAP VALUES IN FITTING EQUATIONS AND CONTOUR $
VERT =  (0.603).YDIM $  WDIM = '0.603).WDIM $
DW = (WB - WT)/WDIM $ KY = (YB - YT)/YDIM $
DX = (XR - XL)/HOR $
DY = (YB - YT)/VERT   $
OUTPUT  CNDATA(XL, XR, YT, YB, RF, CON, LEV(K))$
FORMAT CONDAT(*X VALUE LEFT EDGE OF MAP = *,X9.1,
       *       X VALUE RIGHT EDGE OF MAP =* ,X8.1,
       *       Y VALUE TOP EDGE OF MAP  = *,X8.1,W,
              *Y VALUE BOTTOM EDGE OF MAP = *,X7.1,
       *       REFERENCE CONTOUR VALUE = *,X9.1,
     *         CONTOUR INTERVAL = *,X13.2,W,
              *ELEVATION OF MAP DATUM = *,X11.1,W,W) $
OUTPUT LXDATA(LEX(K), WT, WB, YB, YT, CON) $
FORMAT  LXFMT(*VERTICAL PROFILE PARALLEL TO W-Y PLANE AND INTERSECTING*
          ,* X AXIS AT *,X10.1,W,
          *W VALUE TOP EDGE OF PROFILE =* ,X7.1,B7,
          *W VALUE BOTTOM EDGE OF PROFILE =* ,X7.1,B4,
          *Y VALUE LEFT  EDGE OF PROFILE =* ,X7.1,W,
          *Y VALUE RIGHT EDGE OF PROFILE =* ,X7.1,B10,
          *CONTOUR INTERVAL = *,X6.2,W,W) $
OUTPUT LYDATA(LAY(K), WT, WB, XL, XR, CON) $
FORMAT LYFMT(*VERTICAL PROFILE PARALLEL TO  W-X PLANE AND INTERSECTING*
          ,* Y AXIS AT *,X10.1,W,
          *W VALUE TOP EDGE OF PROFILE =*,X7.1,B7,
          *W VALUE BOTTOM EDGE OF PROFILE =* ,X7.1,B4,
          *X VALUE LEFT EDGE OF PROFILE =*, X7.1,W,
          *X VALUE RIGHT EDGE OF PROFILE =*,X7.1,B10,
          *CONTOUR INTERVAL = *,X6.2,W,W) $
W = 1 $
CALZ1..
I = 1 $        K = 1 $
IF W GEQ 4 $ GO PROFILEX $
IF W EQL 1 $
   BEGIN
      LAB3..
      WRITE ($$ ALPHA, FMTA) $
      FORMAT OHED1(*CONTOURS OF LINEAR TREND VOLUME *,W ) $
      WRITE ($$ ZHED1 ) $
WRITE ($$ CNDATA,CONDAT) $
      BQ = Q(1) + (Q(2).LEV(K)) + (Q(3).XL) $
      BQ1 = Q(3).DX $
      CALZ3..
      BQ2 =(Q(4).(YT+(DY.I))) + BQ $
      FOR A = (1.0,1.0,HAR) $
```

```
        Z(A) = BQ2 + (A.BQ1) $
        GO CONTOUR $
     END $
  IF W EQL 2 $
     BEGIN
        LAB4..
        WRITE ($$ ALPHA,FMTA) $
        FORMAT ZHED2(*CONTOURS OF LINEAR + QUADRATIC TREND VOLUME *,W)$
        WRITE   ($$ ZHED2 ) $
  WRITE ($$ CNDATA,CONDAT) $
        BS1  = S(3).DX      $
        BS2  = S(5).(2XL.DX)  $
        BS3  = S(5).(DX.DX)   $
        BS5  = S(8).(LEV(K).DX) $
        CALZ4..
        BSY  = YT +(DY.I)  $
        BS4  = S(7).(BSY.DX)  $
        BS = S(1) + (S(2).LEV(K)) + (S(3).XL) + (S(4).BSY) +
           (S(5).(XL.XL)) + (S(6).(LEV(K).BSY)) + (S(7).(BSY.XL)) +
           (S(8).(LEV(K).XL)) + (S(9).(LEV(K).LEV(K))) +
           (S(10).(BSY.BSY)) $
        BSA = BS1 + BS2 + BS4 + BS5 $
        FOR A =(1.0,1.0,HAR) $
        Z(A) = BS + (A.BSA) + (BS3.(A.A)) $
        GO CONTOUR $
     END $
  IF W EQL 3 $
     BEGIN
        LAB5..
        WRITE ($$ ALPHA, FMTA) $
        FORMAT ZHED3(*CONTOURS OF LINEAR + QUADRATIC + CUBIC TREND *,
           *VOLUME *,W) $
        WRITE ($$ ZHED3) $
  WRITE ($$ CNDATA,CONDAT) $
        BT1 = F(3).DX     $
        BT2 = F(5).(2XL.DX)   $
        BT3 = F(5).(DX.DX)    $
        BT5 = F(8).(LEV(K).DX) $
        BT6 = F(11).(3XL.(XL.DX))   $
        BT7 = F(11).(3XL.(DX.DX))   $
        BT8 = F(11).(DX.(DX.DX))    $
        BTAA = BT3 + BT7 $
        CALZ5..
        BTY = YT +(DY.I) $
        BT4 = F(7).(BTY.DX)   $
        BT =  F(1) +(F(2).LEV(K)) + (F(3).XL) + (F(4).BTY) +
           (F(5).(XL.XL)) + (F(6).(LEV(K).BTY))+ (F(7).(BTY.XL)) +
           (F(8).(LEV(K).XL)) + (F(9).(LEV(K).LEV(K))) +
           (F(10).(BTY.BTY)) + (F(11).(XL.(XL.XL))) +
           (F(12).(LEV(K).(LEV(K).LEV(K)))) + (F(13).(BTY.(BTY.BTY))) $
        BTA = BT1 + BT2 + BT4 + BT5 + BT6 $
        FOR A =(1.0,1.0,HAR) $
        Z(A) = BT + (A.BTA) + (BTAA.(A.A)) + (BT8.(A.(A.A))) $
        GO CONTOUR $
     END $
  CONTOUR..
  FOR J =(1,1,HOR) $
     BEGIN
        IF Z(J) LSS RF $
           BEGIN
              CV(J) = ARMIN(MOD(FIX((RF-Z(J))/CON),40) +1) $
              GO THERE $
           END $
```

```
        CV(J) = ARPLS(MOD(FIX((Z(J)-RF)/CON),40)+1) $
THERE..  END  $
OUTPUT ODCV(FOR J =(1,1,HOR) $ CV(J))$
FORMAT FTCV(132A1,W) $
WRITE ($$ ODCV,FTCV) $
I = I + 1 $
IF (W EQL 1) AND (I LEQ VERT) $ GO CALZ3 $
IF (W EQL 2) AND (I LEQ VERT) $ GO CALZ4 $
IF (W EQL 3) AND (I LEQ VERT) $ GO CALZ5 $
K = K + 1 $
IF (W EQL 1) AND (K LEQ EL) $ (I = 1 $ GO LAB3) $
IF (W EQL 2) AND (K LEQ EL) $ (I = 1 $ GO LAB4) $
IF (W EQL 3) AND (K LEQ EL) $ (I = 1 $ GO LAB5) $
W = W + 1 $ GO CALZ1 $
PROFILEX..
W = 1 $
CALX1..
I = 1 $ K = 1 $
IF W  GEQ 4 $  GO PROFILEY $
IF W EQL 1 $
    BEGIN
        LABX3..
        WRITE ($$ ALPHA, FMTA) $ WRITE ($$ ZHED1) $
        WRITE ($$ LXDATA, LXFMT) $
        BXA2 = Q(4).KY $
        BXA1 = Q(1) +(Q(2).WT) + (Q(3).LEX(K)) + (Q(4).YB) $
        CALX3..
        BXA3 = BXA1 + (Q(2).(DW.I)) $
        FOR A =(1.0,1.0,YDAM) $
        Z(A) =  BXA3 - (A.BXA2) $
        GO KONTOUR1 $
    END $
IF W EQL 2  $
    BEGIN
        LABX4..
        WRITE ($$ ALPHA, FMTA) $ WRITE ($$ ZHED2) $
        WRITE ($$ LXDATA, LXFMT) $
        BX0 = S(1) +(S(3).LEX(K)) + (S(4).YB) + (S(5).(LEX(K).LEX(K))) +
            (S(7).(LEX(K).YB)) + (S(10).(YB.YB))  $
        BX1 =  S(4).KY  $
        BX3 =  S(7).(LEX(K).KY) $
        BX4 =  S(10).(2YB.KY) $
        BX5 =  S(10).(KY.KY)  $
        BTX1 = S(2) + (S(6).YB) + (S(8).LEX(K)) $
        CALX4..
        BTX  = WT +(DW.I) $
        BX2  =(S(6).(BTX.KY)) $
        BTQ  =(BTX1.BTX) + (S(9).(BTX.BTX)) + BX0 $
        BTX2 = BX1 + BX2 + BX3 + BX4 $
        FOR A =(1.0,1.0,YDAM) $
        Z(A) =-(A.BTX2)+(((A.A).BX5) + BTQ)$
        GO KONTOUR1 $
    END $
IF W EQL 3 $
    BEGIN
        LABX5..
        WRITE ($$ ALPHA, FMTA) $ WRITE ($$ ZHED3) $
        WRITE ($$ LXDATA, LXFMT) $
        BD0  =  F(1) + (F(3).LEX(K)) + (F(4).YB) + (F(5).(LEX(K).LEX(K)))
            + (F(7).(LEX(K).YB)) + (F(10).(YB.YB))  + (F(11).(LEX(K).
            (LEX(K).LEX(K))))+ (F(13).(YB.(YB.YB)))  $
        BD1 = F(2) + (F(6).YB) + (F(8).LEX(K)) $
        BD2 = (F(4).KY) $
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        BD4 = (F(7).(LEX(K).KY))  $
        BD5 =2(F(10).(YB.KY)) $
        BD6 = F(10).(KY.KY) $
        BD7 =3(F(13).(YB.(YB.KY)))$
        BD8 =3(F(13).(YB.(KY.KY)))$
        BD9 =  F(13).(KY.(KY.KY)) $
        CALX5..
        BDX = WT +(DW.I)  $
        BD10 = BD2 + (F(6).(BDX.KY))+  BD4 + BD5  + BD7 $
        BD11 = BD6 + BD8  $
        BD12 =(BD1.BDX)+ BD0 +(F(9).(BDX.BDX)) + (F(12).(BDX.(BDX.BDX)))$
        FOR A =(1.0,1.0,YDAM) $
        Z(A) =  BD12 - (A.BD10) + ((A.A).BD11) - ((A.BD9).(A.A)) $
        GO KONTOUR1 $
    END $
KONTOUR1..
FOR J = (1,1,YDIM) $
    BEGIN
        IF Z(J) LSS RF $
            BEGIN
                CV(J) = ARMIN(MOD(FIX((RF-Z(J))/CON),40) +1) $
                GO  HERE $
            END $
        CV(J) = ARPLS(MOD(FIX((Z(J)-RF)/CON),40)+1) $
 HERE..  END  $
OUTPUT OCDX(FOR J =(1,1, YDIM) $ CV(J)) $
WRITE ($$ OCDX, FTCV) $
I = I + 1 $
IF (W EQL 1) AND (I LEQ WDIM) $ GO CALX3 $
IF (W EQL 2) AND (I LEQ WDIM) $ GO CALX4 $
IF (W EQL 3) AND (I LEQ WDIM) $ GO CALX5 $
K = K + 1 $
IF (W EQL 1) AND (K LEQ LX) $ (I = 1 $ GO LABX3)$
IF (W EQL 2) AND (K LEQ LX) $ (I = 1 $ GO LABX4)$
IF (W EQL 3) AND (K LEQ LX) $ (I = 1 $ GO LABX5)$
W = W + 1 $ GO CALX1 $
PROFILEY..
W = 1 $
CALY1..
I = 1 $ K = 1 $
IF W GEQ 4  $ GO PLOTRESID $
IF W EQL 1 $
    BEGIN
        LABY3..
        WRITE ($$ ALPHA, FMTA) $ WRITE ($$ ZHED1) $
        WRITE ($$ LYDATA, LYFMT) $
        BL1 =  Q(1) +(Q(3).XL) + (Q(4).LAY(K)) $
        BL2 =  Q(3).DX $
        CALY3..
        BL3 =((WT +(DW.I)).Q(2)) + BL1 $
        FOR A = (1.0,1.0,HAR) $
        Z(A) = BL3 +(A.BL2) $
        GO KONTOUR2 $
    END $
IF W EQL 2  $
    BEGIN
        LABY4..
        WRITE ($$ ALPHA, FMTA) $ WRITE ($$ ZHED2) $
        WRITE ($$ LYDATA, LYFMT) $
        BN1 =  S(1) + (S(3).XL) + (S(4).LAY(K)) + (S(5).(XL.XL)) +
          (S(7).(LAY(K).XL)) + (S(10).(LAY(K).LAY(K))) $
        BN2 = S(3).DX    $
        BN3 = S(5).(2.(XL.DX)) $
```

```
        BN4 = S(5).(DX.DX) $
        BN5 = S(7).(LAY(K).DX) $
        BN6 = BN2 + BN3 + BN5 $
        CALY4..
        BTX = WT +(DW.I) $
        BN8 =(S(8).(BTX.DX))+ BN6 $
        BN9 = BN1 + (S(6).(BTX.LAY(K))) + (S(8).(BTX.XL)) +
            (S(9).(BTX.BTX)) + (S(2).BTX) $
        FOR A =(1.0,1.0,HAR) $
        Z(A) = BN9 + (A.BN8) + ((A.A).BN4) $
        GO KONTOUR2 $
     END $
  IF W EQL 3 $
     BEGIN
        LABY5..
        WRITE ($$ ALPHA, FMTA) $ WRITE ($$ ZHED3) $
        WRITE ($$ LYDATA, LYFMT) $
        BRO =  F(1)              + (F(3).XL) + (F(4).LAY(K)) +
            (F(5).(XL.XL)) + (F(7).(LAY(K).XL)) + (F(10).(LAY(K).LAY(K)))
            +(F(11).(XL.(XL.XL))) + (F(13).(LAY(K).(LAY(K).LAY(K)))) $
        BR1 = F(3).DX $
        BR2 =  2(F(5).(XL.DX)) $
        BR3 = F(5).(DX.DX) $
        BR4 = F(7).(LAY(K).DX) $
        BR6 = 3(F(11).(XL.(XL.DX)))  $
        BR7 = 3(F(11).(XL.(DX.DX)))  $
        BR8 = F(11).(DX.(DX.DX)) $
        BR9 = BR1 + BR2 + BR4 + BR6 $
        BR10 = BR3 + BR7  $
        CALY5..
        BDX = WT + (DW.I) $
        BR11 = BR9 + (F(8).(BDX.DX)) $
        BR12=(BDX.( F(2) + (F(6).LAY(K)) + (F(8).XL) + (F(9).BDX) +
           (F(12).(BDX.BDX))))+ BRO $
        FOR A =(1.0,1.0,HAR) $
        Z(A) = BR12 + (A.BR11) + ((A.A).BR10) + ((A.A).(A.BR8)) $
        GO KONTOUR2 $
     END $
  KONTOUR2..
  FOR J =(1,1,HOR) $
     BEGIN
        IF Z(J) LSS RF $
           BEGIN
              CV(J) = ARMIN(MOD(FIX((RF-Z(J))/CON),40) +1) $
              GO THER  $
           END $
        CV(J) = ARPLS(MOD(FIX((Z(J)-RF)/CON),40)+1) $
  THER..   END  $
  OUTPUT ODCY(FOR J =(1,1,HOR) $ CV(J))$
  WRITE ($$ ODCY,FTCV) $
  I = I + 1 $
  IF (W EQL 1) AND (I LEQ WDIM) $ GO CALY3 $
  IF (W EQL 2) AND (I LEQ WDIM) $ GO CALY4 $
  IF (W EQL 3) AND (I LEQ WDIM) $ GO CALY5 $
  K = K + 1 $
  IF (W EQL 1) AND (K LEQ LY) $ (I = 1 $ GO LABY3)$
  IF (W EQL 2) AND (K LEQ LY) $ (I = 1 $ GO LABY4)$
  IF (W EQL 3) AND (K LEQ LY) $ (I = 1 $ GO LABY5)$
  W = W + 1 $ GO CALY1 $
  PLOTRESID..
  IF RESIDOP NEQ 1 $ GO START $
  OUTPUT OUT(FOR I =(1,1,K) $ PRINT(2,I)) $
  FOR I =(1,1,N) $
```

```
    BEGIN
       IY(I) =  X(I,3)/DY  $
       IX(I) =  X(I,2)/DX  $
    END  $
FOR C =(4,2,10)$
BEGIN
FOR FW = (LOW, STEP, HIGH)$
BEGIN       J= 0 $
    FOR I= (1,1,N)$
    BEGIN IF (X(I,1) GEQ FW) AND (X(I,1) LSS(FW+ STEP))$
    BEGIN J = J + 1 $ I2(J) = I $ ENDS  ENDS
N2 = J $
FWSTEP = FW + STEP       $
WRITE (SS ALPHA,FMTA)$ WRITE ($S PLOT, PLFT) $ WRITE($SRESOUT,RESLEV)$
FOR LINE =(1,1,VERT) $
    BEGIN
       K=0$
       FOR I =(1,1,N2)$
           BEGIN
              J = I2(I) $
              IF IY(J) EQL LINE $
               BEGIN
                 K = K + 1 $
                 PRINT(1,K) = IX(J) $
                 PRINT(2,K) =  X(J,C) $
                 DIGITS(K)= 0.4343.LOG((QQQ= ABS(X(J,C))) +(QQQ LSS 1.0
                          ))+ 2 $
               END $
           END $
       FOR I =(2,1,K) $ FOR J =(1,1,I-1)$
           BEGIN
              IF PRINT(1,I) LSS PRINT(1,J) $
                 BEGIN
                    FOR L = 1,2 $
                       BEGIN
                          TEMP = PRINT(L,J) $
                          PRINT(L,J) = PRINT(L,I) $
                          PRINT(L,I) = TEMP $
                       END $
                       TEMP = DIGITS(J)$ DIGITS(J)=DIGITS(I)$
                       DIGITS(I) = TEMP $
                 END $
           END $
       FOR I = (2,1,K) $ PRINT(3,I) = 0 $
       PRINT(3,1) = PRINT(1,1)- DIGITS(1) $
       FOR I = (2,1,K) $
           BEGIN
              IF PRINT(3,I-1) LSS 0 $
                 BEGIN
                    PRINT(3,I) = PRINT(3,I-1) $
                    PRINT(3,I-1) = 0 $
                 ENDS
              PRINT(3,I) = PRINT(1,I) + PRINT(3,I) - PRINT(1,I-1)
                 - DIGITS(I) $
           ENDS
       WRITE ($S OUT, FORM) $
    END $
 END $        ENDS
 GO START $ FINISH $
 1 ((G  7    ()G
 J ,X)*PE,2 9 '76'75'7447 '7X'76 7 77X'75 449' 7448' '4 '' 9' '4 '' 9'
 J7((*P'-    74'G  04'9 '76 75 6 '06705'-4'76 75 74 549
  FINISH $
```


APPENDIX *C*

EXAMPLES OF OUTPUT FROM COMPUTER PROGRAM

Part 1

Example of output from program, listing (1) values of elements in matrix and column vector, (2) equation coefficients, (3) statistical measures of hypersurfaces, (4) hypervolumes and spatially weighted averages of z within hypersurfaces, and (5) table of values for data points, listing original data, and trend and residual values.

4-VARIABLE RELATIONS OIL GRAVITY, WELL DEPTH, GEOG LOC IN SE KANSAS

13 X 13 (X,Y) MATRIX VALUES

2.44, 02	5.69, 03	7.44, 02	9.73, 02	3.51, 03	2.27, 04	3.09, 03	1.48, 04	1.45, 05	5.26, 03	2.05, 04	3.87, 06	3.30, 04
5.69, 03	1.45, 05	1.48, 04	2.27, 04	6.07, 04	5.92, 05	5.67, 04	3.36, 05	3.87, 06	1.21, 05	3.16, 05	1.07, 08	7.45, 05
7.44, 02	1.48, 04	3.51, 03	3.09, 03	2.05, 04	5.67, 04	1.58, 04	6.07, 04	3.36, 05	1.77, 04	1.32, 05	8.20, 06	1.15, 05
9.73, 02	2.27, 04	3.09, 03	5.26, 03	1.58, 04	1.21, 05	1.77, 04	5.67, 04	5.92, 05	3.30, 04	9.67, 04	1.64, 07	2.25, 05
3.51, 03	6.07, 04	2.05, 04	1.58, 04	1.32, 05	2.40, 05	9.67, 04	3.16, 05	1.22, 06	9.73, 04	9.04, 05	2.73, 07	6.69, 05
2.27, 04	5.92, 05	5.67, 04	1.21, 05	2.40, 05	3.18, 06	2.97, 05	1.24, 06	1.64, 07	7.45, 05	1.30, 06	4.73, 08	5.02, 06
3.09, 03	5.67, 04	1.58, 04	1.77, 04	9.67, 04	2.97, 05	9.73, 04	2.40, 05	1.24, 06	1.15, 05	6.45, 05	3.02, 07	8.07, 05
1.48, 04	3.36, 05	6.07, 04	5.67, 04	3.16, 05	1.24, 06	2.40, 05	1.22, 06	8.20, 06	2.97, 05	1.89, 06	2.10, 08	1.60, 06
1.45, 05	3.87, 06	3.36, 05	5.92, 05	1.22, 06	1.64, 07	1.24, 06	8.20, 06	1.07, 08	3.18, 06	5.71, 06	3.03, 09	1.97, 07
5.26, 03	1.21, 05	1.77, 04	3.30, 04	9.73, 04	7.45, 05	1.15, 05	2.97, 05	3.18, 06	2.25, 05	6.26, 05	9.01, 07	1.61, 06
2.05, 04	3.16, 05	1.32, 05	9.67, 04	9.04, 05	1.30, 06	6.45, 05	1.89, 06	5.71, 06	6.26, 05	6.39, 06	1.16, 08	4.44, 06
3.87, 06	1.07, 08	8.20, 06	1.64, 07	2.73, 07	4.73, 08	3.02, 07	2.10, 08	3.03, 09	9.01, 07	1.16, 08	8.78, 10	5.70, 08
3.30, 04	7.45, 05	1.15, 05	2.25, 05	6.69, 05	5.02, 06	8.07, 05	1.60, 06	1.97, 07	1.61, 06	4.44, 06	5.70, 08	1.19, 07

1 X 13 COLUMN VECTOR VALUES

8.96, 03	2.11, 05	2.67, 04	3.56, 04	1.24, 05	8.42, 05	1.09, 05	5.39, 05	5.40, 06	1.93, 05	7.17, 05	1.45, 08	1.21, 06

EQUATION COEFFICIENTS

LINEAR, Z = 36.30596+ .0812557W + -.3449529X + -.0900418Y

LIN + QUAD, Z = 37.90548+ .0662714W + -1.0362317X + -.6831682Y + .0319961 4X2 + -.03304167WY + -.09474352XY

+ .03123953WX + .00094850W2 + .20109156Y2

LIN + QUAD + CUB, Z = 45.20266+ -.9616497W + -.3940345X + -4.3746648Y + -.01570635X2 + -.03085394WY

+ -.07620920XY + .01609525WX + .05903346W2 + 1.16790819Y2 + .00014462X3 + -.00093207W3 + -.07155833Y3

ERROR MEASURE LINEAR TREND SURFACE = 9.86

ERROR MEASURE QUADRATIC TREND SURFACE = 8.82

ERROR MEASURE CUBIC TREND SURFACE = 8.04

PERCENT TOTAL SUM SQUARES LINEAR SURFACE = 31.77

PERCENT TOTAL SUM SQUARES QUADRATIC SURFACE = 49.69

PERCENT TOTAL SUM SQUARES CUBIC SURFACE = 62.96

SUM OF SQUARES DUE LINEAR COPPONENT = 330624.23

SUM OF SQUARED DEVIATIONS FROM LINEAR = 2395.34

SUM OF SQUARES DUE LINEAR + QUADRATIC COMPONENT = 330876.85

SUM OF SQUARES DUE TO QUADRATIC ALONE = 252.62

SUM OF SQUARED DEVIATIONS FROM LINEAR + QUADRATIC = 2142.68

SUM OF SQUARES DUE LINEAR+QUADRATIC+CUBIC = 331065.94

SUM OF SQUARED DEVIATIONS FROM LINEAR+QUADRATIC+CUBIC = 1953.58

SUM OF SQUARES DUE CUBIC ALONE = 189.09

VOLUME WITHIN LINEAR SURFACE = 96579.17

VOLUME WITHIN LIN+QUAD SURFACE = 97826.64

VOLUME WITHIN LIN+QUAD+CUB SURFACE = 99533.01

ARITH. MEAN Z, = SUM OF Z VALUES/ N = 36.79

AVERAGE Z VALUE, LINFAR SURFACE = 35.87

AVERAGE Z VALUE, LIN+QUAD SURFACE = 36.33

AVERAGE Z VALUE, LIN+QUAD+CUB SURFACE = 36.96

VCL OF BLOCK IN CUBED UNITS 2692.80

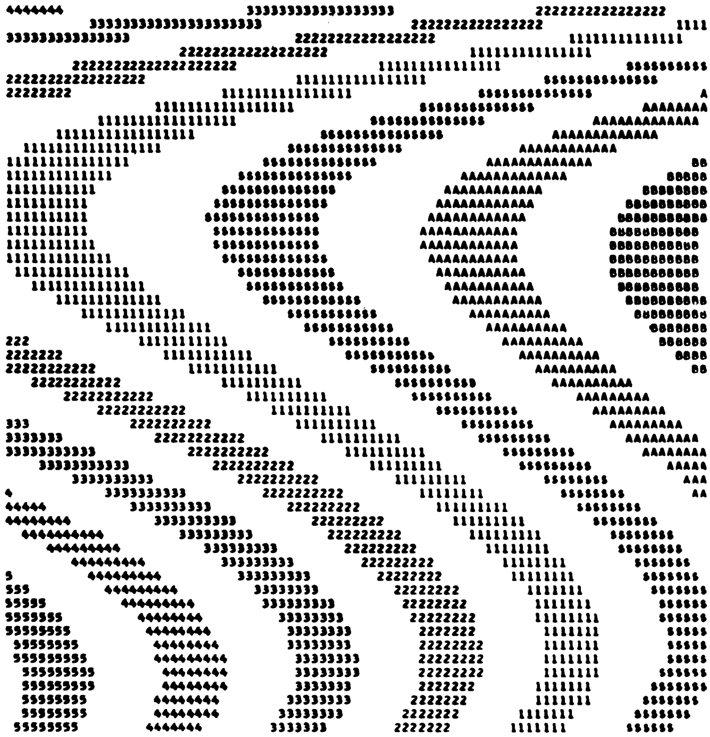
WCOORD	XCOORD	YCOORD	Z-VALUE	1ST-TRD	1ST-RSD	2ND-TRD	2ND-RSD	3RD-TRD	3RD-RSD
19.0	6.1	.7	39.4	35.7	3.7	37.2	2.2	37.5	1.9
16.1	7.9	.3	37.4	34.9	2.5	37.0	.4	37.6	-.2
17.8	6.0	1.5	34.8	35.7	-.9	36.3	-1.5	35.3	-.5
20.0	6.1	1.5	34.3	35.7	-1.4	36.3	-2.0	35.3	-1.0
17.9	7.8	.7	36.6	35.0	1.6	36.9	-.3	36.5	.0
15.2	8.0	.9	35.0	34.7	.3	35.6	-.6	34.8	.2
17.3	7.5	.5	38.7	35.1	3.6	37.0	1.7	37.3	1.4
19.5	7.4	.5	41.8	35.0	6.8	36.7	5.1	37.0	4.8
16.0	7.0	.1	41.4	35.3	6.1	38.0	3.4	39.6	1.8
25.0	4.3	1.7	38.5	36.7	1.8	37.4	1.1	37.0	1.5
25.0	4.3	2.0	30.9	36.7	-5.8	37.1	-6.2	36.4	-5.5
24.0	4.5	1.6	37.2	36.6	.6	37.3	-.1	36.9	.3
23.5	4.5	.9	39.6	36.6	3.0	38.2	1.4	38.7	.9
23.7	5.0	.8	40.7	36.4	4.3	38.4	2.3	39.1	1.6
24.5	4.7	1.4	40.3	36.5	3.8	37.7	2.6	37.4	2.9
22.0	5.3	.5	41.0	36.2	4.8	38.4	2.6	39.6	1.4
21.0	5.5	1.7	33.6	36.0	-2.4	36.4	-2.8	35.4	-1.8
22.5	5.4	.2	39.2	36.3	2.9	39.1	.0	41.2	-2.0
23.0	5.5	.1	40.2	36.3	3.9	39.5	.7	41.9	-1.7
19.1	6.2	2.3	30.3	35.5	-5.2	35.2	-4.9	33.6	-3.3
7.8	6.4	3.3	30.8	34.4	-3.6	32.0	-1.2	31.8	-1.0
19.5	6.4	3.0	26.6	35.4	-8.8	34.6	-8.0	33.1	-6.5
12.3	6.4	3.4	35.7	34.8	.9	32.9	2.8	31.6	4.1
34.7	.1	8.5	39.5	38.3	1.2	40.3	-.8	38.0	1.5
37.6	.1	8.7	41.6	38.5	3.1	40.2	1.4	36.0	5.6

Part 2

Examples of contours drawn on sides of block intersecting third-degree hypersurface. Reference contour line is formed by edge of band of $ signs facing band of A's.

```
CRUDE OIL GRAVITY RELATED TO DEPTH AND LOCATION IN SOUTHEAST KANSAS

CONTOURS OF LINEAR + QUADRATIC + CUBIC TREND VOLUME
X VALUE LEFT EDGE OF MAP =        .0      X VALUE RIGHT EDGE OF MAP =     8.5       Y VALUE TOP EDGE OF MAP  =       .0
Y VALUE BOTTOM EDGE OF MAP =     8.8      REFERENCE CONTOUR VALUE =      34.0        CONTOUR INTERVAL =             .80
ELEVATION OF MAP DATUM =         6.0
```



```
CRUDE OIL GRAVITY RELATED TO DEPTH AND LOCATION IN SOUTHEAST KANSAS

CONTOURS OF LINEAR + QUADRATIC + CUBIC TREND VOLUME
VERTICAL PROFILE PARALLEL TO W-Y PLANE AND INTERSECTING X AXIS AT        8.5
W VALUE TOP EDGE OF PROFILE =    6.0       W VALUE BOTTOM EDGE OF PROFILE =   36.0    Y VALUE LEFT  EDGE OF PROFILE =    8.8
Y VALUE RIGHT EDGE OF PROFILE =     .0          CONTOUR INTERVAL =    .80
```

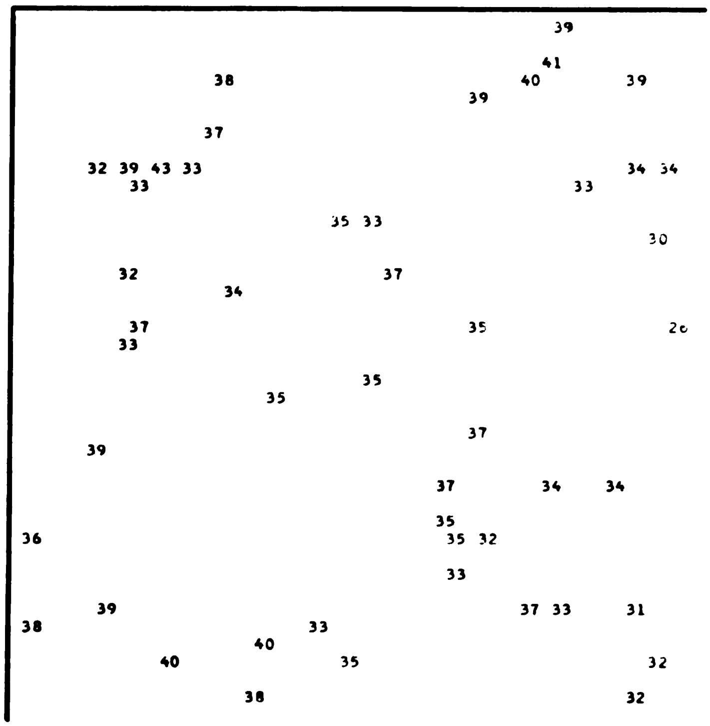
Part 3

Examples of slice maps where data values have been plotted by computer. Map below contains original data values and map on p. 58 contains second-degree residual values. Lines were added by hand to show left and top boundaries of each map. Origin is in upper left corner of each map where lines intersect.

CRUDE OIL GRAVITY RELATED TO DEPTH AND LOCATION IN SOUTHEAST KANSAS

Z VALUES ORIG DATA

LOWER LEVEL OF SLICE = 24.00 UPPER LEVEL OF SLICE = 18.00



CRUDE OIL GRAVITY RELATED TO DEPTH AND LOCATION IN SOUTHEAST KANSAS

2ND ORDER RESIDUAL

LOWER LEVEL OF SLICE = 24.00 UPPER LEVEL OF SLICE = 18.00